August 31, 2010

Mathematics Proficiency Exam Fall 2010

1. This exam has five questions. You are required to answer **any four** of the five questions. Each question is worth 25 points. You have three hours to write the exam.

2. This is a closed book exam. You are not allowed to use calculators in the exam.

3. If you find any question ambiguous, explain your confusion and make whatever assumptions you think are necessary to answer the question. Clearly state any additional assumptions you make.

GOOD LUCK !

1. [(a) =12, (b) = 13]
Consider the system of equations given by:

$$Ax = b$$

where A is an $m \times n$ matrix and b is a non-zero vector in $\mathbb{R}^m$. Denote by r the rank of A. Assume that the rank of the augmented matrix A_b is equal to r, and that $r < n$.

(a) Define the set S as follows:

$$S = \{x \in \mathbb{R}^n : Ax = 0\}$$

Show that there exists a set of $(n-r)$ linearly independent vectors $\{y^1, ..., y^{n-r}\}$, with $y^i \in S$ for $i \in \{1, ..., n-r\}$, such that every $x \in S$ can be expressed as a linear combination of the set of vectors $\{y^1, ..., y^{n-r}\}$.

(b) Define the set T as follows:

$$T = \{x \in \mathbb{R}^n : Ax = b\}$$

Show that there exists a set of $(n - r + 1)$ linearly independent vectors $\{z^1, ..., z^{n-r+1}\}$, with $z^i \in T$ for $i \in \{1, ..., n-r\}$, such that given any $x \in T$, one can find numbers $t_1, ..., t_{n-r+1}$ such that $(t_1 + \cdots + t_{n-r+1}) = 1$, and:

$$x = t_1 z^1 + \cdots + t_{n-r+1} z^{n-r+1}$$

2. [(a) = 13, (b) = 12]

Let f be a continuous function from $\mathbb{R}^2_+$ to $\mathbb{R}$, which is twice continuously differentiable on $\mathbb{R}^2_{++}$. Suppose, for every $i \in \{1,2\}$, we have $D_i f(x) > 0$ for all $x \in \mathbb{R}^2_{++}$. Further, suppose that:

$$\det \begin{bmatrix} 0 & D_1 f(x) & D_2 f(x) \\ D_1 f(x) & D_{11} f(x) & D_{12} f(x) \\ D_2 f(x) & D_{21} f(x) & D_{22} f(x) \end{bmatrix} > 0$$

for all $x \in \mathbb{R}^2_{++}$.

Let $(\bar{p}_1, \bar{p}_2, \bar{x}_1, \bar{x}_2) \in \mathbb{R}^4_{++}$ satisfy the following system of equations:

$$\left.\begin{array}{l} \bar{p}_2 D_1 f(\bar{x}_1, \bar{x}_2) - \bar{p}_1 D_2 f(\bar{x}_1, \bar{x}_2) = 0 \\ f(\bar{x}_1, \bar{x}_2) - 1 = 0 \end{array}\right\}$$

(a) Use the implicit function theorem to show that there exists an open set $A \subset \mathbb{R}^2_{++}$ containing $(\bar{p}_1, \bar{p}_2)$, and an open set $B \subset \mathbb{R}^2_{++}$ containing $(\bar{x}_1, \bar{x}_2)$, and a unique function $h : A \to B$ such that for all $(p_1, p_2) \in A$,

$$\begin{array}{l} p_2 D_1 f(h^1(p_1, p_2), h^2(p_1, p_2)) - p_1 D_2 f(h^1(p_1, p_2), h^2(p_1, p_2)) = 0 \\ f(h^1(p_1, p_2), h^2(p_1, p_2)) - 1 = 0 \end{array}$$

and $(h^1(\bar{p}_1, \bar{p}_2), h^2(\bar{p}_1, \bar{p}_2)) = (\bar{x}_1, \bar{x}_2)$. [Here, h^1 and h^2 denote the two component functions of h].

(b) Note that h is continuously differentiable on A, and show that $D_1 h^1(\bar{p}_1, \bar{p}_2) < 0$.

3. [(a) = 15, (b) = 10]

Let a, b, c, A be arbitrary positive real numbers. Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \textit{Maximize} & x_2 + A\ln(1+x_1) \\ \textit{subject to} & ax_1 + bx_2 \leq c \\ \textit{and} & (x_1, x_2) \in \mathbb{R}^2_+ \end{array}\right\} (P)$$

(a) Use the Kuhn-Tucker sufficiency theorem to obtain the optimal solution to (P) in terms of the parameters a, b, c, A.

(b) Keeping the parameters b, c, A fixed, draw a graph to show how the optimal solution to (P) of x_1 varies with the parameter a.

4. [(a) = 8, (b) = 9, (c) = 8]

Let S be an open interval of $\mathbb{R}$, such that $[0, 1] \subset S$. Let f be a function from S to $\mathbb{R}$ which is increasing and continuously differentiable on S. Assume that (i) $f(0) = 0$, (ii) $f'(0) > f'(1)$, and (iii) $f(1/2) < [f(1)/2]$.

Consider the following constrained optimization problem:

$$\left.\begin{array}{ll} Max & f(x) + f(y) \\ subject\ to & x + y \leq 1 \\ and & (x, y) \geq 0 \end{array}\right\} (Q)$$

(a) Use Weierstrass theorem to show that problem (Q) has a solution.

(b) Let $(\hat{x}, \hat{y})$ be any solution to problem (Q). Apply the Arrow-Hurwicz-Uzawa necessity theorem to show that (i) $\hat{x} > 0, \hat{y} > 0$, and (ii) $\hat{x} \neq \hat{y}$.

(c) Using (b), show that there are at least three distinct solutions of $x \in (0, 1)$ for the equation:

$$f'(x) = f'(1 - x)$$

5. [(a) = 7, (b)(i) = 10, (b)(ii) = 8]

(a) Let B be an arbitrary non-negative number. Consider the function:

$$f(x, y) = (x + B)y \text{ for all } (x, y) \in \mathbb{R}^2_+$$

Show that f is quasi-concave on $\mathbb{R}^2_+$ but f is not concave on $\mathbb{R}^2_+$.

(b) Let a, b, c be arbitrary positive real numbers. For every $t \in [0, 1]$, consider the constrained maximization problem:

$$\left.\begin{array}{ccc} Maximize & & [x + (2tc/a)]\, y \\ subject & to & ax + by \leq (1 - t)\, c \\ and & & (x, y) \in \mathbb{R}^2_+ \end{array}\right\} (R)$$

(i) Use the Arrow-Enthoven sufficiency theorem to obtain the value function for problem (R). Denote the value function by $V(t)$.

(ii) Let E be the set defined by:

$$E = \{t \in [0, 1] : V(0) \leq V(t)\}$$

Find the maximum value of t which belongs to the set E, showing your procedure clearly.