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On the Dynamics of Community Development*

Abstract

We present a dynamic political economy model of community development. A community invests in a local public good. It can grow, with new housing supplied by competitive developers. To finance investment, the community can tax residents and issue debt. In each period, fiscal decisions are made by current residents. The community's initial wealth determines how it develops. High wealth leads to rapid development. Low wealth leads to gradual development, fueled by community wealth accumulation. In the long run the community can be too large or too small. Development may proceed too slowly. Public good provision is efficient at all times.

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1 Introduction

A long tradition in public and urban economics sees the role of communities as facilitating the joint consumption of local public goods (schools, roads, parks, libraries, shopping and entertainment areas, etc). Accordingly, a large literature studies the formation and development of communities which independently finance and provide their local public goods. The focus of this paper is on the dynamic development of such communities. In order for development to occur, it is simultaneously necessary for residents to invest in local public goods for new residents and for developers to build housing for them to live in. This paper asks: if public investment and financing decisions are made sequentially by current residents and new housing is provided by competitive developers, how would we expect a community to develop? In particular, would we expect development to proceed efficiently and, if not, what are the nature of the distortions?

From a policy perspective, this question is interesting in light of the long-standing debate about whether it is better to decentralize decisions regarding infrastructure investment and development to local communities or to have these decisions controlled by higher levels of government. An understanding of the performance of decentralized regimes in which local governments control and finance all their investment is key to this debate.

From a theoretical perspective, what makes the question intriguing is that the market for new development is influenced by two conflicting forces. On the one hand, potential residents and developers do not take into account the positive cost-sharing externality conferred on existing residents by their building houses in the community. On the other, potential residents and developers have an incentive to free-ride on a community's assets in their building decision. Thus, a community with a good stock of local public goods and low levels of public debt will be more attractive for developers than one with low public assets. These two considerations mean that a community with a low stock of public assets is likely to attract too little development, while a community with good assets is likely to attract too much. However, the level of a community's public assets is endogenous and depends on its past fiscal decisions. Specifically, it will be determined by the community's investment decisions and how it has chosen to finance these investments (i.e., with debt or taxes). These decisions will be taken by residents with their impact on development very much in mind. Moreover, the extent of development will itself feedback to influence the nature of future community fiscal decisions.

Shedding light on the question requires a dynamic model that permits analysis of the evolution of a community's public assets and the interaction between these assets and new development. The model studied in this paper is an infinite horizon, partial equilibrium model of a single community. At any point in time, the community is characterized by its stock of local public good, its debt, and its housing stock. At the beginning of each period, the current residents of the community choose how much to invest in the public good and how to finance this investment. Investment can be financed both by issuing debt and levying taxes. Taxes are paid by all households residing in the community at the end of the period. Following these policy choices, the housing market opens and determines the price of housing and quantity of new construction. On the demand side, there is a pool of potential residents. Households in the pool obtain a uniform utility from living outside the community, but have heterogeneous willingnesses to pay to live in it. This generates a downward sloping demand curve for the community's housing. There is turnover, with households entering and exiting the pool each period. Supply comes from two sources: current residents who leave the pool of potential residents and have to sell their homes, and new construction. Building land is not scarce and new houses are supplied by competitive developers. Once built, houses are infinitely durable.

When choosing how much to invest and how to finance this investment, current residents anticipate how their decisions will impact both the housing market and the policy decisions of future residents. They care about the housing market because it determines the value of their homes and the number of residents of the community. The value of their homes is relevant because, if they have to leave the community, they need to sell their houses. The number of residents matters because of the cost-sharing externality created by the public good. Current residents care about the policy decisions of future residents because these determine future home values and public good surplus.

The sequential interaction of the different cohorts of residents gives rise to a dynamic game. Following the usual approach in dynamic political economy, we look for a Markov equilibrium in which residents' policy decisions and housing market outcomes just depend on the three state variables - the stock of the public good, debt, and the housing stock. In the equilibrium we find, the community's initial stock of public assets determines how it develops. The key variable is the community's wealth, which is the value of its stock of public good less its outstanding debt. There are three ranges of initial wealth - high, medium, and low - that are associated with distinct patterns of development. In the high range, development occurs rapidly and the community's wealth remains constant. In the remaining two ranges, development occurs gradually and the community's wealth accumulates as it develops. The low range is distinguished from the medium by a period of community wealth building that precedes new construction.

When the community's initial wealth is in the high range, in the initial period, residents invest in the public good, anticipating an influx of new residents, and the market provides new construction. The residents finance the increase in the stock of the public good entirely with debt, so that the community's wealth remains constant. Thereafter, there is no further new construction and investment occurs only to replace depreciated units of the public good. Taxes are set to pay for maintaining the public good and servicing the debt.

In the medium range, residents also invest in the public good in the initial period, anticipating an increase in population, and the market provides new construction. However, residents build the community's wealth by financing some of the investment with taxation. The associated higher taxes depress development relative to what would happen under the path associated with the high range, but result in the community being in a stronger position at the beginning of the next period. The same thing happens in subsequent periods. Thus, the market provides new construction in each period and residents build the community's wealth by financing some of the additional public good provided for the larger population with taxation. This wealth building paves the way for more development in the next period. The size of the community increases gradually and continually over time, converging asymptotically to a steady state level.

In the low range, no new construction takes place in the initial period. Residents tax themselves to build the community's wealth and developers are deterred by the high taxes. Wealth building is achieved by investing in the public good and/or reducing the community's debt, and paves the way for future development. Thereafter, the size of the community grows to the same steady state level as in the medium wealth case. Convergence either proceeds gradually or takes place in one period, depending on the size of the initial housing stock.

Whatever the community's initial wealth, development proceeds inefficiently. In the high wealth range, the equilibrium size of the community is either too small or too large, depending on the extent of its initial wealth. In the medium and low wealth ranges, the steady state size of the community is too small. In addition, the development path exhibits a further type of distortion: namely, inefficient delay. On a more positive note, in all three cases, some development occurs. This is the case even when the community's initial wealth is sufficiently low to make it highly unattractive to developers. Moreover, at all times, the level of public good provided is optimal given the number of residents. This reflects the fact that residents can control wealth by changing the community's debt, so there is never a need to distort the public good for this purpose.

The results are distinctive in highlighting the importance of a community's stock of public assets in determining its development when new construction is provided by the market. They also provide a completely novel account of the role of community wealth accumulation in development. The fundamental economic incentive to accumulate wealth comes from the desire to attract more households to share the costs of the public good. As noted earlier, a community with better public assets is more attractive to potential residents and developers. On the other hand, to accumulate wealth, a community needs to levy higher taxes which imposes costs on current residents and makes the community less attractive to developers in the short-run. Moreover, the future benefits of wealth accumulation necessarily accrue partially to future residents and thus current residents do not fully appropriate these benefits. These forces mean that wealth accumulation will be under-supplied by residents. Nonetheless, wealth accumulation does occur when the community's initial wealth is in the medium or low ranges, and enables the community to increase the amount of development it experiences. The equilibrium describes how this happens and reveals the economic forces that sustain it.

The analysis offers three new lessons for the policy debate about the desirability of decentralizing decisions regarding infrastructure investment and development to local communities. The first is that in a decentralized regime in which new construction is determined by the market, a community may need to build wealth to achieve a socially optimal size. Importantly, it is not sufficient for a community to just borrow to improve its local public goods: tax financed investment is necessary. The second lesson is that, while such wealth building will occur to some extent, it will not happen sufficiently and it may take place only gradually. This reflects the fact that current residents do not appropriate the full benefits from their investment. It means that a community will end up under-sized and that development will occur too slowly. The final lesson is that a community with access to debt can be expected to provide the socially optimal amount of public goods for its residents. Under or over-provision of public infrastructure should not be a concern.

The organization of the remainder of the paper is as follows. Section 2 discusses related literature. Section 3 introduces the model and Section 4 characterizes socially optimal development. Section 5 defines equilibrium development. Section 6 explains our strategy for finding equilibrium and describes the equilibrium that we uncover. Section 7 analyzes how development proceeds in this equilibrium. Section 8 compares equilibrium and optimal development. Section 9 concludes with suggestions for future work.

2 Related literature

The paper relates to various literatures in public economics, urban economics, and political economy. Most relevant is the literature already mentioned studying communities which independently finance and provide their local public goods (see Ross and Yinger 1999 and Wildasin 1986 for reviews). The most prominent strand of this literature analyzes how households who differ in incomes or tastes or both, locate across different communities under the assumption that, once located, households collectively choose taxes and public good levels for their communities and optimize their housing consumption (see, for example, Epple, Filimon, and Romer 1984, Fernandez and Rogerson 1998, and Rose-Ackerman 1979). The number of communities and the housing supplies in each community are exogenous and the focus is on the existence of equilibrium and on whether the population is segmented across communities in the way envisioned by Tiebout (1956). Our analysis shares with this literature the view that the role of communities is to facilitate the joint consumption of local public goods. It also has in common the assumption that fiscal decisions are made by residents. It differs in that its focus is on the dynamic development of a single community rather than the allocation of households across multiple communities.

Another strand of this literature takes a different approach to the community formation process by assuming that communities are formed by monopoly developers (see, for example, Henderson 1985 and Sonstelie and Portney 1978). These developers acquire land, build housing, and provide public goods with the aim of attracting residents and making profits. They face a pool of potential residents who must be provided with at least some target utility level to be induced to locate in their communities. Developers' profits are the revenue from selling property less the costs of construction and public good provision. This literature studies the efficiency of public good provision, housing levels, and the allocation of households across communities. Our analysis shares with this literature the idea that community decisions are made strategically with an eye on how they will attract potential residents. It differs in the sense that decisions are made by residents, rather than monopoly developers. Moreover, these residents have only imperfect control over the level of new construction, which is supplied by competitive developers.

While there is recognition of its importance, the dynamic development of communities has received limited attention in this literature. Henderson (1980) studies the developer's problem in a two period, single community setting. There are two groups of potential residents: those around in the first period and those arriving in the second. The developer sells homes in the initial period to the period one potential residents, sells further homes in period two to the second group, and provides public goods in both periods. Period one residents remain in the community for both periods and public good provision is financed by a property tax in each period. Henderson shows that if the developer can commit to policies at the beginning of period one, the home sizes and public good levels provided in both periods will be efficient.[1] In our analysis, the residents are the decision-makers and decision-making is sequential, so there is no commitment. Moreover, homes come in one variety and, while the set of potential residents is changing, its size is constant across periods. Finally, and most significantly, because the public good is durable and the community can borrow, the community has assets.

The urban economics literature explores reasons for agglomeration other than the collective consumption of local public goods. The famous monocentric city model originating in the works of Alonso (1964), Mills (1967), and Muth (1969), assumes that households agglomerate to be near their place of work in the city center. In the influential work of Krugman (1991), producers of differentiated products and worker/consumers with tastes for variety agglomerate to be able to produce, work, and trade with lower transactions costs. Some work in this literature develops dynamic models of city development. For example, a number of authors have studied development in the monocentric city model under the assumption that population is growing exogenously (for a review of this work, see Brueckner 2000). Of particular interest is the work of Henderson and Venables (2009) who consider a model in which new population arrives continuously and cities form sequentially. Each city has the structure of the monocentric model, but the model allows for other forms of agglomeration externalities. As in our model, housing, once constructed, is infinitely durable. The planning solution involves one city after another being created and filled to optimal size with arriving citizens. A decentralized equilibrium is studied in which construction, as in our model, is chosen by competitive builders and there are no local city governments to coordinate development. In this equilibrium, cities could be too small or too large. The paper then studies an equilibrium in which profit maximizing city developers offer subsidies to residents to live in their cities. Developers commit to these subsidies before their cities are formed and subsidies vary over the course of city development. This equilibrium replicates the planning solution.

This paper differs from this literature in its focus on the development of a small community

[1] Henderson notes a tension between the interests of the developer and the period one residents. At the beginning of period two, the developer would like to shift more of the second period tax burden to initial residents and hence extract higher sales revenues from period two residents. He points out that this tension creates a potential time inconsistency problem. Our model also displays a time inconsistency problem, in the sense that current residents prefer that future residents choose different policies than they do.

in which people live, rather than a city where people also produce and trade. More importantly, what is distinctive about this paper is its modelling of community policy-making (specifically, sequential decision-making by home-owning residents) and its incorporation of community assets (a durable public good and debt). There are a number of interesting consequences of this set-up. One is that gradual growth emerges organically rather than being driven by exogenous changes in the environment, such as population growth. Another is the interaction between community assets and development, particularly the community wealth accumulation that fuels development. A third is the inefficiency that arises in the size and pace of development. This differs from the literature, which emphasizes the ability of local governments to achieve efficient solutions.[2]

Turning back to public economics, the paper relates to the literature on the capitalization of local government assets (schools, infrastructure, debt, pension obligations, etc) into housing prices stemming from Oates (1969). The structure of the model implies that such capitalization is operational, but incomplete. In any period, the supply curve of housing looks like an inverted L. It is vertical up to a price equal to the construction cost with a quantity equal to the current housing stock, and then becomes horizontal, as new construction is added. This implies that an increase in the community's assets will not be capitalized into housing prices if demand prior to the increase is already sufficient to bring forth new construction. The increase in demand will just be met by an increase in new construction. However, if even after the increase, demand is insufficient to spur new construction, then capitalization will be just as in a model with fixed housing supply. This incomplete capitalization has interesting implications for residents' incentives to invest and borrow.

Another related public economics literature, is that on the political economy of public good provision. This literature explores how public good provision is determined in different political settings (see, for example, Baron 1996, Bergstrom 1979, Lizzeri and Persico 2001, and Romer and Rosenthal 1979). Most of the work, including the state and local literature just discussed, focuses on the provision of static public goods, which must be provided anew each period. In practice, many important public goods are durable, lasting for many years and depreciating relatively slowly. Understanding the political provision of such goods is more challenging, because of their durable nature. Recent work has studied the provision of such goods in a variety of settings.[3] This paper

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[2] It should be noted that our residents do not have access to new construction subsidies or taxes. Nonetheless, as we discuss below, these will not resolve the inefficiencies the model identifies. Indeed, they may well make matters worse.

[3] Battaglini and Coate (2007), Battaglini, Nunnari, and Palfrey (2012), and LeBlanc, Snyder, and Tripathi (2000) study legislative provision, Barseghyan and Coate (2014) and Coate and Ma (2017) study bureaucratic provision,

adds to this strand of the literature by studying the provision of a local durable public good by residents in a growing community with population turnover. An interesting prediction of the model is that, despite population turnover and incomplete capitalization, investment is always efficient given the size of the community.[4] We argue that this reflects the assumption that the government can borrow.

Also related, is the literature on the political economy of public debt. A large literature studies the accumulation of debt at the national government level in various political settings (for a review see Alesina and Passalacqua 2015). The key focus has been on understanding why debt accumulation may be excessive. Less attention has been paid to the debt of local governments. A notable feature of such debt is that, once a resident leaves the locality, he/she ceases to have any responsibility for it. One theme in the literature is that this may result in the costs of debt not being fully bourne by the issuing residents. On the other hand, in a world in which the supply of housing is fixed, local government debt should be capitalized into the price of housing, putting the full burden of debt on the issuing residents even if they leave (see, for example, Daly 1969). As just discussed, in the model of this paper, capitalization is incomplete. Nonetheless, the fact that higher debt levels are capitalized into housing prices if new construction is not undertaken, prevents debt from being abused. In particular, residents do not increase debt and use it to finance tax cuts for themselves because this would deter development and lead to a fall in the value of their homes. In equilibrium, the wealth of the community either stays constant or grows over time and debt plays a key role in both allowing the community to develop and to provide public goods efficiently.[5]

A final related public economics literature is that on clubs, particularly club dynamics. Roberts (2015) considers a club that provides a public good and shares the costs of provision among its

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and Conley, Driskill, and Wang (2013) and Schultz and Sjostrom (2001) study provision in a multi-community setting.

[4] In a community with a fixed housing supply and population turnover, capitalization provides incentives for residents to make efficient public good investment decisions despite the fact that they will not be around to enjoy all the benefits. See Brueckner and Joo (1991) and Conley, Driskill, and Wang (2013) for further discussion of the incentives provided by capitalization for public good provision.

[5] The role played by debt in the model differs from the role commonly discussed in the literature. The usual argument is that because residents may leave their community, they will underinvest in durable public goods if they must finance investment with taxes. Debt financing counteracts this distortion because mobility also implies that residents do not bear the full burden of debt issued (for a formal analysis see Schultz and Sjostrom 2001). This logic underlies the so-called "golden rule" that prescribes that local governments pay for non-durable goods and services with tax revenues and use debt to finance investment in durables (for discussion and analysis see Bassetto with Sargent 2006). In our model, no golden rule is imposed and residents finance investment with a mix of debt and taxes. Debt allows the community to accumulate wealth in a way that does not require distorting public good investment. Wealth accumulation is necessary to attract new residents who are needed to share the costs of public good provision.

members. Potential members differ in the strength of their public good preferences. Each period, the existing members (who are those with the strongest public good preferences), must decide how many new members to admit. All potential members would like to be admitted, and, once admitted, remain in the club indefinitely. After new members have been admitted, the membership decides on a level of public good to provide. The public good is non-durable and club decisions are made via majority rule. The existing members face a trade-off: if they admit more members, this reduces the per-member cost of the public good, but it also changes the provision level because new members have weaker preferences. This trade-off gives rise to interesting equilibrium membership dynamics. While there are many differences, the model of this paper shares with Robert's work that the key rationale for the community is the sharing of the costs of a public good, and, that the dynamic structure of community decision-making involves existing residents choosing policies which determine next period's residents.[6] Moreover, as we point out in the conclusion, the model and techniques of this paper should be useful for analyzing the dynamic development of clubs.

Finally, the paper contributes to the fast growing literature developing and analyzing infinite horizon political economy models of policy-making with rational, forward-looking decision makers.[7] It is well recognized that many interesting issues arise from recognizing the dynamic linkage of policies across periods. Such linkages arise directly, as with public investment or debt, or indirectly because current policy choices impact citizens' private investment decisions. The model studied in this paper is distinctive in featuring both state variables directly controlled by the voters (debt and the stock of public good) and a state variable determined by the market (the housing stock). It also features a changing group of decision-makers, as the size of the community is growing. Despite these complications, we are able to provide something very close to a closed form solution of the model.

[6] The differences are first, that there is no conflict among residents concerning policy: all residents, irrespective of how long they have lived in the community, have the same policy preferences. Second, that new residents are free to move to the community if they are willing to buy a house, so residents control membership of the community only indirectly via the impact of their policy choices on the housing market. Third, that existing residents may need to leave the community, in which case they sell their houses. Fourth, that the public good is durable and the cost of investment can be shifted intertemporally via debt. Fifth, that the problem faced by the existing residents is that they need to attract new residents and this requires offering a higher level of public good surplus.

[7] Examples of this style of work are Azzimonti (2011), Battaglini and Coate (2008), Bowen, Chen, and Eraslan (2015), Coate and Morris (1999), Hassler, Rodriguez Mora, Storesletten, and Zilibotti (2003), and Krusell and Rios-Rull (1999). Examples which share the focus of this paper on state and local public finance are Barseghyan and Coate (2016) and Brinkman, Coen-Pirani, and Sieg (2016).

3 The model

Consider a community such as a small town or village.[8] This community can be thought of as one of a number in a particular geographic area. The time horizon is infinite and periods are indexed by $t = 0, ..., \infty$. There is a pool of potential residents of the community of size 1. These can be thought of as households who for exogenous reasons (employment opportunities, family ties, etc) need to live in the area in which the community is located and are potentially open to living in the community. Potential residents are characterized by their desire to live in the community (as opposed to an alternative community in the area) which is measured by the preference parameter θ. This desire, for example, may be determined by a household's idiosyncratic reaction to the community's natural amenities. The preference parameter takes on values between 0 and $\overline{\theta}$, and the fraction of potential residents with preference below $\theta \in [0, \overline{\theta}]$ is $\theta/\overline{\theta}$. Reflecting the fact that households' circumstances change over time, in each period new households join the pool of potential residents and old ones leave. The probability that a household currently a potential resident will be one in the subsequent period is $\mu \in (0, 1)$. Thus, in each period, a fraction $1 - \mu$ of households leave the pool and are replaced by an equal number of new ones.

The only way to live in the community is to own a house. The community has sufficient land to accommodate housing for all the potential residents. Moreover, the only use for land is building houses.[9] Houses are infinitely durable and the cost of building a new one is C.[10] Housing is supplied by competitive developers. The stock of houses at the beginning of a period is denoted by H and the stock at the beginning of the next period by H'. New construction is therefore $H' - H$. A stock of housing H can accommodate a fraction H of the pool of potential residents. The initial housing stock is denoted H_0.[11]

The community provides a durable local public good which depreciates at rate $\delta \in (0, 1)$. The good costs c per unit. The stock at the beginning of a period is denoted by g and the stock at the

[8] The underlying economic model is a single community version of that presented by Barseghyan and Coate (2016), amended to include infinitely durable housing, a congestible durable local public good, and debt. A similar model is employed by Coate and Ma (2017) in their critique of using housing price changes to evaluate public investment, although they assume a fixed housing stock and no debt.

[9] We could alternatively assume that land not used for housing has a constant productivity in agricultural use.

[10] The assumption of infinitely durable housing is common in the urban economics literature and is justified by the fact that buildings in developed countries display considerable longevity. See Brueckner (2000) for more discussion of the different modelling assumptions used in the literature.

[11] It is important that H_0 be positive, so the community has residents. If this were not the case, there would be nobody to choose the period 0 policies. This assumption distinguishes the paper from the work of Henderson (1980) who considers the development of an empty community by a developer. In future work, it would be interesting to add an additional initial development stage managed by developers to the model of this paper.

beginning of the next period by g'. Given depreciation, the public good level available during the period is $g'/(1-\delta)$ and investment is $g'/(1-\delta)-g$.[12] The initial public good level is denoted g_0.

When living in the community, households have preferences defined over the public good and consumption. A household with preference parameter θ and consumption x obtains a period payoff of $\theta+x+B([g'/(1-\delta)]\,/\,(H')^{\alpha})$ from living in the community if the public good level is $g'/(1-\delta)$ and the number of households is H'.[13] The public good benefit function B is increasing, smooth, strictly concave, and satisfies the limit conditions that $lim_{z\searrow 0}B'(z)=\infty$ and $lim_{z\nearrow\infty}B'(z)=0$. The parameter α measures the congestibility of the public good and belongs to the interval $[0,1]$. The smaller is α, the closer the good is to a pure public good. The higher is α, the closer the good is to a publicly provided private good. When not living in the community, a household's per period payoff is $\underline{u}$.[14] Households discount future payoffs at rate β and can borrow and save at rate $\rho=1/\beta-1$. This assumption means that households are indifferent to the intertemporal allocation of their consumption. Each household in the pool receives an exogenous income stream, the present value of which is sufficient to pay taxes and purchase housing in the community.[15]

A competitive housing market operates in each period. Demand comes from new households moving into the community, while supply comes from owners leaving the community and new construction. The price of houses is denoted P. This price can fall below the construction cost C when demand at C is less than the stock H.

The community can also borrow and save at rate ρ. The community's debt level at the beginning of a period is denoted by b and the level at the beginning of the next period by b'. The community levies a tax T which is paid by all households who reside in the community at the end of the period. The community's budget constraint is therefore

$$(1+\rho)b+c\left(\frac{g'}{1-\delta}-g\right)=b'+H'T. \tag{1}$$

The left hand side is government spending and consists of debt repayment and investment. The right hand side is government revenues and consists of new borrowing and tax revenues. The community's initial debt level is denoted b_0.

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[12] Disinvestment is possible, so that $g'/(1-\delta)$ can be smaller than g. This assumption is made for the purposes of tractability. While unrealistic for durable public goods like roads, the reader can be assured that disinvestment does not occur in the equilibrium we study.

[13] We do not distinguish between the size of the housing stock and the number of residents since these will be the same in equilibrium.

[14] Note that $\underline{u}$ is both the per period payoff of living in one of the other communities in the geographic area if a household is in the pool and the payoff from living outside the area when a household leaves the pool.

[15] The assumption that utility is linear in consumption means that there are no income effects, so it is not necessary to be specific about the income distribution.

The timing of the model is as follows. Each period, the community starts with a public good level g, a debt level b, and a stock of houses H. At the beginning of the period, the existing residents choose the level of investment in the public good, how to adjust the community's debt position, and how much tax to levy. This determines g', b', and T. Then, households who were in the pool of potential residents in the previous period learn whether they will be remaining and new households join. Those in the pool decide whether to live in the community and existing residents no longer in the pool prepare to leave it. The housing market opens and the equilibrium price P is determined along with new construction or, equivalently, next period's housing stock H'. New residents buy houses and move into the community and old ones sell up and leave. Residents enjoy public good benefits $B([g'/(1-\delta)]\,/\,(H')^{\alpha})$ and pay taxes T. The policies must satisfy the community's budget constraint. Next period begins with the state (g', b', H').

4 Optimal community development

It is straightforward to characterize the development plan that would be optimal for a Utilitarian planner. Such a planner maximizes the discounted sum of the aggregate payoffs of the different pools of potential residents. The assumption that utility is linear in consumption, implies that the planner is indifferent between transfers of consumption both between households in the same pool and across different pools. Accordingly, there is no loss of generality in simply assuming that, in any period, the cost of new construction and investment is financed by lump-sum taxation of the pool of potential residents.

The planner's problem can be posed recursively. Given initial stocks of public good and housing (g, H), the planner chooses investment and new construction or, equivalently, next period's public good and housing stock (g', H'). The planner must respect the feasibility constraints

$$g' \geq 0 \;\&\; H' \geq H. \tag{2}$$

The planner will allocate the households in the pool with the highest θ to the H' houses. Given that θ is uniformly distributed on $[0, \overline{\theta}]$, this implies that households in the interval $[(1-H')\,\overline{\theta}, \overline{\theta}]$ will be assigned to live in the community. Accordingly, the planner's problem is

$$U(g,H) = \max_{(g',H')} \left\{ \begin{array}{c} \int_{(1-H')\overline{\theta}}^{\overline{\theta}} \theta \frac{d\theta}{\overline{\theta}} + H'B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right) + \underline{u}\,(1-H') - C(H'-H) \\ -c(g'/(1-\delta) - g) + \beta U(g', H') \\ s.t.\ (2) \end{array} \right\}. \tag{3}$$

The first two terms in the objective function represent the benefits received by the households assigned to the community, while the third term represents the benefits to those not so assigned. The fourth and fifth terms represent, respectively, the costs of new construction and investment. The final term is the continuation value.

Observe that g enters the value function linearly, so that $\partial U(g,H)/\partial g$ is just equal to c. Given this, it is straightforward to verify that g' must equal $(1-\delta)g^o(H')$ where $g^o(H)$ satisfies the dynamic Samuelson rule

$$H^{1-\alpha}B'\left(\frac{g^o}{H^\alpha}\right) = c\left[1-\beta(1-\delta)\right]. \tag{4}$$

The left hand side measures the per-period social benefit of an additional unit of public good and the right hand side the per-period cost. The latter reflects the fact that a fraction $1-\delta$ of a unit purchased this period will be available for use next period.

To characterize the optimal level of housing in a way that makes it comparable with what happens in equilibrium, it is convenient to introduce the function $S(H)$ which represents per-resident *optimized public good surplus*, defined as

$$S(H) \equiv B\left(\frac{g^o(H)}{H^\alpha}\right) - \frac{cg^o(H)(1-\beta(1-\delta))}{H}. \tag{5}$$

This surplus is the difference between the public good benefits enjoyed by each resident at the optimal level and the per-resident cost of this level computed using the per-period marginal cost from (4). Using this function, we can substitute out the public good and rewrite the planner's problem as follows:

$$U(H) = \max_{H'} \left\{ \begin{array}{c} \int_{(1-H')\overline{\theta}}^{\overline{\theta}} \theta \frac{d\theta}{\overline{\theta}} + H'S(H') + \underline{u}\,(1-H') - C(H'-H) \\ +c(1-\delta)(g^o(H) - \beta g^o(H')) + \beta U(H') \\ s.t.\ H' \geq H \end{array} \right\}. \tag{6}$$

Maximizing with respect to H' and using the envelope condition that $U'(H')$ is equal to $C + c(1-\delta)dg^o(H')/dH$ reveals that the optimal level of housing H^o satisfies the first order condition[16]

$$(1-H^o)\,\overline{\theta} + S(H^o) + H^o S'(H^o) - C(1-\beta) = \underline{u}. \tag{7}$$

The left hand side represents the net social benefit from assigning an additional household to the community. The first term is the preference of the marginal household for living in the community;

[16] This assumes that the constraint $H' \geq H$ is not binding.

the second, the optimized public good surplus accruing to the marginal household; the third, the impact of adding the household on the public good surpluses of the other residents; and the fourth, the per-period cost of an additional house. The optimal housing level is such that this net social benefit is just equal to the benefit the household receives when not residing in the community, $\underline{u}$. Note that

$$HS'(H) = \frac{(1-\alpha)cg^o(H)(1-\beta(1-\delta))}{H}, \tag{8}$$

so that the impact on other residents' public good surpluses of adding a household is always positive provided that α is less than 1. This reflects the benefits of sharing the costs of the public good.

We impose the following assumption to make sure that the planner's problem is well-behaved.

Assumption 1 (i) *For all* $H \in [H_0, 1]$

$$-\overline{\theta} + 2S'(H) + HS''(H) < 0.$$

(ii)

$$(1-H_0)\,\overline{\theta} + S(H_0) + H_0 S'(H_0) - C(1-\beta) > \underline{u} > S(1) + S'(1) - C(1-\beta).$$

Part (i) of the assumption implies that net social benefit from assigning an additional household to the community is decreasing in the number of households. Part (ii) implies that this net social benefit exceeds $\underline{u}$ at the initial population H_0 but falls below it when the entire pool lives in the community. Together, the two parts imply that there exists a unique solution to the first order condition (7) that lies between H_0 and 1. This solution unambiguously defines the optimal housing level. We may therefore conclude:

Proposition 1 *Under Assumption 1, the optimal community development plan is to construct* $H^o - H_0$ *new houses in period* 0 *and invest in* $g^o(H^o) - g_0$ *units of the public good. Thereafter, no more housing should be constructed and the public good level should be maintained at* $g^o(H^o)$.

There are three main points to take away about the optimal plan. First, development occurs immediately. Second, the public good satisfies the dynamic Samuelson rule. Third, the size of the community balances the net social benefit of an additional household to the payoff households get from living elsewhere.

5 Equilibrium community development defined

We now study equilibrium community development. This section explains what an equilibrium in the model is. As explained above, the timing of the model is first that the existing residents choose fiscal policies, and then the housing market determines new construction and the price of housing. Obviously, when residents choose policies they will anticipate how they impact the housing market. Rather than deriving the relationship between the housing market equilibrium and the policies, and then analyzing the optimal policies, it is easier to think of residents as directly choosing the housing price and new construction along with the policies, but subject to the constraint that their choice be consistent with housing market equilibrium. Thus, given any initial state (g, b, H), we will assume that residents choose (g', b', T, H', P) to maximize their expected payoffs subject to the various constraints on policy and the requirements of housing market equilibrium. These payoffs and requirements will now be explained.

Housing demand At the beginning of any period, households fall into two groups: those who resided in the community in the previous period and those who did not, but could in the current period. Households in the first group own homes, while the second group do not. Households in the first group who leave the pool sell their houses and obtain a continuation payoff of

$$P + \frac{u}{1-\beta}. \tag{9}$$

The remaining households in the first group and all those in the second must decide whether to live in the community. Formally, they make a location decision $l \in \{0, 1\}$, where $l = 1$ means that they live in the community. This decision will depend on their preference parameter θ, current and future housing prices, and expected public good provision and taxes. Since selling a house and moving is costless, there is no loss of generality in assuming that all households sell their property at the beginning of any period.[17] This makes each household's location decision independent of its property ownership state. It also means that the only future consequences of the current location choice is through the price of housing in the next period.

To make this more precise, let $V_\theta(g', b', H')$ denote the equilibrium expected lifetime payoff a household with preference parameter θ obtains at the beginning of a period in which the initial state is (g', b', H') and it belongs to the pool but does not own a house. In addition, let $P(g', b', H')$ denote the equilibrium price of housing when the state is (g', b', H'). Then, in a period in which the initial state is (g, b, H), the price of housing is P, and the household anticipates $g'/(1-\delta)$ units of the public good to be provided, H' households to live in the community, and to pay a tax of T,

[17] It should be stressed that this is just a convenient way of understanding the household decision problem. The equilibrium we study is perfectly consistent with the assumption that the only households selling their homes are those who plan to leave the community.

its decision problem can be written as

$$\max_{l\in\{0,1\}}\left\{\begin{array}{c} l\left(\theta+B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-P-T+\beta P(g',b',H')\right) \\ +(1-l)\underline{u}+\beta\left(\mu V_{\theta}(g',b',H')+(1-\mu)\frac{\underline{u}}{1-\beta}\right) \end{array}\right\}. \tag{10}$$

Notice that the payoff the household gets when locating in the community reflects the without loss of generality assumption that it sells the house it buys at the beginning of the next period. Inspecting this problem, it is clear that the household will choose to reside in the community if and only if

$$\theta+B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-P-T+\beta P(g',b',H')\geq\underline{u}. \tag{11}$$

The left hand side of this inequality represents the per-period payoff from locating in the community, assuming that the household buys a house at the beginning of the period and sells it the next. This payoff depends on the preference parameter θ, public good surplus, and current and future housing prices. The right hand side represents the per-period payoff from living elsewhere.

Housing market equilibrium Continue to assume that the initial state is (g,b,H) and that households anticipate that $g'/(1-\delta)$ units of the public good will be provided, that H' households will live in the community, that the tax will be T, and that next period's equilibrium price of housing will be $P(g',b',H')$. Then, given (11) and the fact that household preferences are uniformly distributed over $[0,\overline{\theta}]$, the equilibrium price of housing P in the current period must satisfy the market clearing condition

$$H'=1-\frac{\underline{u}-\left(B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-P-T+\beta P(g',b',H')\right)}{\overline{\theta}}. \tag{12}$$

This implies that the equilibrium price is

$$P=(1-H')\overline{\theta}+B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-T+\beta P(g',b',H')-\underline{u}. \tag{13}$$

Moreover, because the supply of new construction is perfectly elastic at a price equal to the construction cost C, it must also be the case that

$$P\leq C\ (=\text{if } H'>H). \tag{14}$$

Given that next period's housing price is described by the function $P(g',b',H')$, any policy (g',b',T,H',P) which satisfies the budget constraint (1) and the feasibility constraints (2), is consistent with housing market equilibrium if and only if (13) and (14) are satisfied.[18]

[18] This formulation implicitly assumes that P can, in principle, be negative. This is unrealistic because, in reality, residents can simply abandon their houses and leave the community if they so choose. However, imposing the

Policy choice Given any initial state (g, b, H), existing residents will desire a policy (g', b', T, H', P) that maximizes their payoffs subject to the community budget constraint (1), the feasibility constraints (2), and the market equilibrium constraints (13) and (14). While residents differ in their desires to live in the community θ, they will have identical preferences over policies and hence there is no collective choice problem to resolve. To understand this, note that given the initial housing stock H, existing residents will have preferences in the interval $[(1-H)\overline{\theta}, \overline{\theta}]$.[19] The objective function of such a type θ resident can be written as:

$$(1-\mu)\left[P+\frac{u}{1-\beta}\right]+\mu\left[\begin{array}{c}\theta+B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-T+\beta P(g',b',H') \\ +\beta\left(\mu V_{\theta}(g',b',H')+(1-\mu)\frac{u}{1-\beta}\right)\end{array}\right]. \tag{15}$$

This reflects the fact that, with probability $1-\mu$, the resident leaves the pool and sells its house, and, with probability μ, it remains in the pool and continues to locate in the community. We know that a resident remaining in the pool will stay in the community because the supply of housing can only expand and the market will allocate housing to those with the highest θ. Iterating this logic implies that for all household types θ in the interval $[(1-H)\overline{\theta}, \overline{\theta}]$, we have that

$$\mu V_{\theta}(g',b',H')+(1-\mu)\frac{u}{1-\beta}=V(g',b',H')-P(g',b',H')+\frac{\theta\mu}{1-\beta\mu}, \tag{16}$$

where $V(g', b', H')$ is defined to be the lifetime payoff of a household with preference parameter $\theta = 0$ at the beginning of a period in which: i) the initial state is (g', b', H'), ii) the household owns a house in the community but does not know whether it will remain in the pool, and iii) the household is constrained to live in the community as long as it remains in the pool. Substituting (16) into (15), we see that θ enters only as an additive constant and thus all existing residents have identical preferences over policies.

In light of this discussion and using the notation $V(g', b', H')$ just introduced, the residents' policy problem can be written as

$$\max_{(g',b',T,H',P)}\left\{\begin{array}{c}(1-\mu)\left[P+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-T+\beta V(g',b',H')\right] \\ s.t.\ (1),\ (2),\ (13),\ \&\ (14).\end{array}\right\} \tag{17}$$

constraint that the equilibrium housing price must be non-negative creates some additional complications that are inessential. In particular, it requires that we introduce a community debt limit. Such a limit is needed to prevent current residents from borrowing a large amount, using it to finance transfers to residents, and then abandoning the community and its debt the next period.

[19] In periods $t = 1, ..., \infty$ this follows from the fact that, in equilibrium, the households with the highest preference for living in the community purchase houses in the community in the previous period. We assume that this condition also characterizes the initial distribution of residents in period 0.

Equilibrium definition An *equilibrium* consists of an investment rule $g'(g, b, H)$, a borrowing rule $b'(g, b, H)$, a tax rule $T(g, b, H)$, a new construction rule $H'(g, b, H)$, a price rule $P(g, b, H)$, and a value function $V(g, b, H)$ satisfying two conditions. First, for all states (g, b, H), the policies solve problem (17) given that the continuation value is described by $V(g', b', H')$ and the future housing price by $P(g', b', H')$. Second, for all (g, b, H), the value function satisfies the equality

$$V(g,b,H) = (1-\mu)\left[P(\cdot) + \frac{u}{1-\beta}\right] + \mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right) - T(\cdot) + \beta V(g'(\cdot), b'(\cdot), H'(\cdot))\right], \tag{18}$$

where $g'(\cdot)$ denotes the policy $g'(g, b, H)$, etc.

6 Finding equilibrium

The strategy for finding equilibrium in this type of model is "guess and verify". The problem here lies in knowing what to guess. The model does not appear to have close cousins in the literature from which insights may be gleaned. To develop intuition for what might happen, we first consider the development plan that would be optimal for the initial residents of the community (i.e., those owning houses at the beginning of period 0). We then consider the time consistency of this plan, to understand in what ways it would need to be modified to survive sequential decision-making. With this information, we formulate a "guess" for an equilibrium and verify analytically that this conjectured equilibrium is indeed an equilibrium. Finally, we discuss the existence of the conjectured equilibrium. Readers just interested in seeing what happens in this equilibrium can skip this section and go straight to Section 7.

6.1 The initial residents' optimal plan

Suppose that the initial residents could commit the community to following a complete development plan $\{g_{t+1}, b_{t+1}, T_t, H_{t+1}, P_t\}_{t=0}^{\infty}$. Here g_{t+1} denotes the level of the public good at the beginning of period $t+1$, b_{t+1} the level of debt at the beginning of period $t+1$, etc. Their optimal plan would maximize the objective function

$$\sum_{t=0}^{\infty} (\mu\beta)^t \left\{(1-\mu)\left[P_t + \frac{u}{1-\beta}\right] + \mu\left[B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - T_t\right]\right\}, \tag{19}$$

subject to in each period t satisfying the budget constraint, the feasibility constraints, and the constraints of market equilibrium.[20]

[20] We also need to add the standard transversality condition that $\lim_{t\to\infty} \beta^t b_t = 0$ to rule out the initial residents operating a Ponzi scheme.

Before we can describe the solution to this problem, we need a little more notation and two additional assumptions. First, let W denote the community's wealth, defined to be the value of the public good less the value of outstanding debt (i.e., $W = cg - (1+\rho)b$). Wealth at the beginning of period t is denoted W_t. Next, for all housing levels in the interval $[H_0, 1]$, let $\mathcal{W}(H)$ be the wealth level implicitly defined by the equality

$$(1-H)\,\overline{\theta} + S(H) + \frac{(1-\beta)W}{H} - C(1-\beta) = \underline{u}. \tag{20}$$

To interpret this wealth level, note that $S(H)+(1-\beta)W/H$ is the public good surplus a household would enjoy if the community had H residents, a wealth level W, and provides the efficient public good level, financing provision so as to keep wealth constant at W.[21] Accordingly, the expression on the left hand side of the equality, represents the per-period benefit that the marginal home buyer would obtain from living in the community under these conditions, assuming the price of housing is constant at C. It follows that $\mathcal{W}(H)$ is the wealth the community needs to attract a population of size H, assuming it provides the efficient public good level and finances provision so as to keep wealth constant.

The first assumption we need is that the community's initial housing stock H_0 satisfy:

Assumption 2

$$\overline{u} + H_0\overline{\theta} > (1-H_0)\,\overline{\theta} + S(H_0) + H_0 S'(H_0) - C(1-\beta) > \underline{u} + H_0\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right).$$

Assumption 2, when combined with Assumption 1, implies that for all housing levels in the interval $[H_0, 1]$, $\mathcal{W}(H)$ is increasing. Thus, the community needs a higher wealth level to attract a larger population. This is intuitive, because households with lower preferences need to be induced to live in the community. This requires offering them more public good surplus, which necessitates lower taxes and thus a higher level of community wealth.[22] Given that $\mathcal{W}(H)$ is increasing on $[H_0, 1]$, it has a well-defined inverse, which we denote $\mathcal{H}(W)$. This function tells us the population the community can attract when it has wealth W. If the community's initial wealth W_0 is such

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[21] Public good surplus if the community has H residents and provides the efficient level of the public good is $B\left(g^o(H)/H^\alpha\right) - T$. Using (1) and the fact that $cg+(1+\rho)b$ is equal to W, the tax will equal $\left[cg^o(H) - W - b'\right]/H$. If the community's wealth is constant at W, this means that $c(1-\delta)g^o(H)+(1+\rho)b'$ is equal to W. Solving this equation for b' and using the fact that $1+\rho$ is equal to $1/\beta$, we can write the tax as $\left[cg^o(H)(1-\beta(1-\delta)) - W(1-\beta)\right]/H$. The claim now follows from the definition of $S(H)$ in (5).

[22] This rules out the possibility that the benefits of sharing the costs of the public good are sufficiently high that, even taking into account the lower desire to live in the community, a larger population actually requires a lower level of wealth to support a market clearing price of C. From (7) this cannot happen for housing levels near the optimal level, but, in general, there is no reason to suppose that it cannot happen at lower housing levels. This is what Assumption 2 rules out.

that $\mathcal{H}(W_0)$ is less than its initial housing stock H_0, it must build wealth in order to develop. By contrast, if $\mathcal{H}(W_0)$ exceeds H_0, development is possible without wealth accumulation.

The second assumption concerns the community's initial wealth W_0:

Assumption 3 *The community's initial public good level g_0 and debt level b_0 are such that*

$$\mathcal{W}(H^o) - H_0 C < W_0 < \mathcal{W}(1).$$

This assumption bounds W_0. The bounds simply ensure that the community neither starts out with so much wealth that everyone wants to live in it, nor is so indebted that no one wants to live in it.

We can now describe the initial residents' optimal plan.

Proposition 2 *If Assumptions 1-3 are satisfied, there exist wealth levels $W^*(H_0)$ and $W_n(H_0)$, satisfying $\mathcal{W}(H_0) < W^*(H_0) < W_n(H_0)$, such that under the initial residents' optimal plan:*
(i) If $W_0 \geq W^*(H_0)$, the community invests in $g^o(\mathcal{H}(W_0)) - g_0$ units of the public good in period 0 and the market provides $\mathcal{H}(W_0) - H_0$ new houses. The community finances investment so as to keep its wealth constant, meaning that all but $c\delta g^o(\mathcal{H}(W_0))$ is financed with debt. Thereafter, the public good is maintained at $g^o(\mathcal{H}(W_0))$ and no more housing is provided. Taxes are set so that wealth remains at W_0. Throughout, the price of housing is C.
(ii) If $W_0 < W^*(H_0)$, the community invests in $g^o(H_0) - g_0$ units of the public good in period 0 and the market provides no new houses. The community chooses debt and taxes so that its wealth increases to $W_n(H_0)$. In period 1, the community invests in $g^o(\mathcal{H}(W_n(H_0))) - (1-\delta)g^o(H_0)$ units of the public good and the market provides $\mathcal{H}(W_n(H_0)) - H_0$ new houses. Investment is financed so as to keep wealth constant, implying that all but $c\delta g^o(\mathcal{H}(W_n(H_0)))$ is financed with debt. Thereafter, the public good is maintained at $g^o(\mathcal{H}(W_n(H_0)))$ and no more housing is provided. Taxes are set so that wealth remains at $W_n(H_0)$. The price of housing is less than C in period 0 and C thereafter.

Thus, the initial residents' optimal plan takes one of two forms. Under the first, anticipating an increase in population, residents invest in the public good in the initial period, and the market provides new construction. The residents finance the increase in the stock of the public good entirely with debt, keeping the community's wealth constant. Thereafter, there is no further development. Under the second form, the community builds wealth in the initial period and no development takes place. The motivation for building wealth is to spur development in the next period. Thereafter, things follow the pattern of the first form. The optimal plan takes the first form when the community has high initial wealth and the second when it has low initial wealth. Here, high and low are defined relative to the endogenous threshold wealth level $W^*(H_0)$.

The proposition tells us that the threshold wealth level $W^*(H_0)$ exceeds $\mathcal{W}(H_0)$. This implies that when the community's initial wealth W_0 lies between $\mathcal{W}(H_0)$ and $W^*(H_0)$, the initial residents choose to build wealth even though development is possible without wealth building (since W_0 being larger than $\mathcal{W}(H_0)$ implies that $\mathcal{H}(W_0)$ exceeds H_0). They do this, because the sacrifice in the initial period in terms of higher taxes and a smaller population, is compensated by the benefits of a larger population in the future.

The proof of this proposition is in Appendix 1, but it is helpful to sketch the key steps here. The first is to note that if the market equilibrium condition (13) is satisfied in each period, the initial residents' objective function will equal

$$P_0 + \mu\overline{\theta} \sum_{t=0}^{\infty} (\mu\beta)^t (H_{t+1} - 1) + \frac{u}{1-\beta}. \tag{21}$$

Thus, housing prices have a direct impact on the objective function only in the initial period. Prices in all subsequent periods wash out. Moreover, all else equal, the initial residents prefer to have future housing stocks as high as possible. This reflects that, when the size of the community is determined by the market, higher housing stocks correspond to greater surplus from living in the community.

The second step is to note that we can assume with no loss of generality that the price of housing is equal to C in all periods other than the initial period. To understand why, imagine that, say, P_1 were less than C. Then it must be the case that there is no new construction in period 1, so that H_2 equals H_1. The market equilibrium condition (13) tells us that P_1 will depend on the period 1 tax. Moreover, the government's budget constraint (1) tells us that this tax will depend positively on the community's period 1 debt b_1. Now suppose that in period 0 taxes were increased to reduce b_1 sufficiently to raise P_1 to C, with population held constant at H_1. The key point to note is that this will not change the housing market equilibrium in period 0. In the market equilibrium condition (13), the increase in taxes this period will be perfectly compensated by an increase in the value of housing next period. In this way, the time path of debt and taxes can be adjusted to make P_1 equal to C without impacting P_0 or the time path of housing. Given (21), this adjustment has no impact on the initial residents' payoff.

The third step is to establish that, in all periods t, the initial residents always provide the optimal public good level for the anticipated number of residents, so that g_{t+1} equals $(1-\delta)g^o(H_{t+1})$.

From (21), the initial residents desire larger housing stocks and these are achieved by creating surplus for residents. Given that the community can use debt to transfer surplus between different cohorts of residents, there is never a reason to distort the public good.

Using the second and third steps, an inter-temporal budget constraint can be created for the community by summing up the sequence of budget constraints (1) and using the market equilibrium conditions (13) to substitute for each period's tax. This inter-temporal budget constraint can be written as

$$\frac{(P_0 - C)H_1}{1-\beta} + \sum_{t=0}^{\infty} \beta^t \mathcal{W}(H_{t+1}) = \frac{W_0}{1-\beta}. \tag{22}$$

This formulation is insightful in illustrating how the community is constrained in the population it can attract by its initial wealth and that this constraint can be relaxed at the cost of reducing P_0 below C. The market equilibrium condition for period 0 implies that

$$P_0 = C - \frac{[(1-\beta)\mathcal{W}(H_1) - (W_0 - \beta W_1)]}{H_1}, \tag{23}$$

so that reducing P_0 below C corresponds to leaving the community with a higher wealth level in period 1. The initial residents' problem then amounts to choosing P_0, W_1, and a sequence of housing levels $\{H_t\}$ to maximize (21) subject to (22), the constraint that housing levels must be non-decreasing, and the period 0 market equilibrium constraints.

The next step is to establish that H_t is constant after period 2, implying that new construction takes place only in periods 0 or 1. This reflects two complementary considerations. The first is the different discount rates in the objective function (21) and the inter-temporal budget constraint (22). The benefits from living in the community in future periods are relevant to the initial residents only if they stay in the community, which implies $\mu\beta$ discounting in the objective function. The budget constraint has to be satisfied regardless of who resides in the community, and, since the community borrows and lends at the market rate $1 + \rho = 1/\beta$, this implies β discounting in the inter-temporal budget constraint. The second consideration is that the wealth the community needs to attract a population of size H, $\mathcal{W}(H)$, is convex under Assumption 1. Thus, the cost of attracting additional households is higher the greater the population, so there is no point in delaying development to reduce these costs.

Given this result, the intertemporal budget constraint (22) and the period 0 market equilibrium condition (23) imply that H_t must equal $\mathcal{H}(W_1)$ for all t beyond period 2. Since (23) also tells us that the period 0 housing price just depends on (W_1, H_1), the initial residents' problem then boils down to just choosing (W_1, H_1). The final step determines this optimal choice. The two options are to develop in period 0 or to build wealth and delay development until period 1. The former involves choosing (W_1, H_1) equal to $(W_0, \mathcal{H}(W_0))$ and the latter involves setting it equal to $(W_n(H_0), H_0)$, where $W_n(H_0)$ is the optimal amount of wealth to build up. Developing in period 0 dominates if and only if the community's initial wealth W_0 exceeds a threshold level $W^*(H_0)$.

6.2 Time consistency of the initial residents' optimal plan

Using standard terminology, the initial residents' optimal plan $\{g_{t+1}, b_{t+1}, T_t, H_{t+1}, P_t\}_{t=0}^{\infty}$ is *time consistent* if, for all $t \geq 1$, $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ is an optimal plan for those residents in the community at the beginning of period t, given the initial condition (g_t, b_t, H_t). To assess time consistency, we need to understand what optimal plans for future residents look like. The optimal plan for the period t residents will solve the same problem as for the period 0 residents, except that the community's wealth and housing stock will be (W_t, H_t) rather than (W_0, H_0). Given Proposition 2, H_t will be at least as big as H_0 and W_t will be at least as big as W_0.

Proposition 3 *If Assumptions 1-3 are satisfied, there exist wealth levels $W^*(H_t)$ and $W_n(H_t)$, satisfying $\mathcal{W}(H_t) \leq W^*(H_t) \leq W_n(H_t)$, such that the period t residents' optimal plan has the form described in Proposition 2. Moreover, the functions $W^*(H_t)$ and $W_n(H_t)$ are continuous and increasing on $[H_0, 1]$, and there exists a housing level $H^s \in (H_0, H^o)$ such that $\mathcal{W}(H_t) < W^*(H_t) < W_n(H_t)$ for all $H_t \in [H_0, H^s)$ and $W^*(H_t) = W_n(H_t) = \mathcal{W}(H_t)$ for all $H_t \in [H^s, 1]$.*

The only qualitative difference between the optimal plans of the initial and period t residents arises when H_t exceeds H^s. In this case, the threshold wealth level $W^*(H_t)$ is equal to $\mathcal{W}(H_t)$ rather than exceeding it. This means that if community wealth is such that development is feasible without wealth accumulation (i.e., $\mathcal{H}(W_t)$ exceeds H_t), then this is what the period t residents will choose. Moreover, $W_n(H_t)$ is equal to the threshold (and hence $\mathcal{W}(H_t)$) rather than exceeding it. $W_n(H_t)$ is the amount of wealth the period t residents would build up to if the community has wealth below the threshold. There is no cost to increasing wealth to $\mathcal{W}(H_t)$, since this will just increase future housing prices to C and will not change the future size of the community. By the logic of the previous sub-section, there will be no impact on the period t housing price. Increasing wealth above $\mathcal{W}(H_t)$ will impact period t residents' payoffs as it will reduce the period t housing price. However, it has the benefit of increasing the future size of the community. The fact that $W_n(H_t)$ is equal to $\mathcal{W}(H_t)$ when H_t exceeds H^s means the period t residents are not willing to bear the costs in terms of the period t housing price to expand population when the community is bigger than H^s.

Intuitively, the housing level H^s identified in Proposition 3 is the smallest housing stock H such that, if the community is endowed with wealth $\mathcal{W}(H)$, it will never choose to increase taxes and build wealth to attract new residents. The existence of such a housing level reflects the fact that the incentives to accumulate wealth are decreasing in the size of the community. This stems from the convexity of the function $\mathcal{W}(H)$ which makes the marginal cost of attracting households higher the greater the population. Formally, H^s is implicitly defined by the equation

$$(1-H)\,\overline{\theta} + S(H) + HS'(H) - C(1-\beta) = \underline{u} + H\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right). \qquad (24)$$

Assumptions 1 and 2 imply that H^s is well-defined and lies in the interval (H_0, H^o).[23]

Combining Propositions 2 and 3, we can now establish:

Proposition 4 *If Assumptions 1-3 are satisfied, the initial residents' optimal plan is time consistent if and only if* $W_0 \geq \mathcal{W}(H^s)$.

To see why the "if" part of the Proposition is true, note that if W_0 exceeds $\mathcal{W}(H^s)$, then the initial residents' optimal plan implies that H_1 exceeds H^s and that W_1 equals $\mathcal{W}(H_1)$. Thereafter, housing and wealth are supposed to remain constant. This is time consistent because if the period t residents have a housing stock H_t exceeding H^s and a wealth level $\mathcal{W}(H_t)$, they will have no incentive to increase taxes and build wealth to attract new residents. They will simply keep wealth and, hence the future housing stock, constant.

For the "only if" part, if W_0 is less than $\mathcal{W}(H^s)$, then W_0 could either exceed or be smaller than the threshold $W^*(H_0)$. In the former case, new construction occurs in period 0 under the initial residents' optimal plan and (W_1, H_1) will equal $(W_0, \mathcal{H}(W_0))$. Thereafter housing and wealth remain constant. However, H_1 will be less than H^s and W_1 will equal $\mathcal{W}(H_1)$ and hence be strictly less than $W^*(H_1)$. This means that the period 1 residents will want to increase taxes and build wealth to attract new residents. In the latter case, new construction does not occur until period 1 and it is the period 2 residents who want to deviate. Under the initial residents' optimal plan, at the beginning of period 2, (W_2, H_2) will equal $(W_n(H_0), \mathcal{H}(W_n(H_0)))$ and thereafter housing and wealth are supposed to remain constant. However, H_2 will be less than H^s and W_2 will equal $\mathcal{W}(H_2)$, which means that the period 2 residents will want to increase taxes and build wealth.

[23] The second inequality in Assumption 2 implies that the left hand side of equation (24) exceeds the right hand side at housing level H_0. The right hand side is increasing in H and Assumption 1(i) implies that the left hand side is decreasing in H. Moreover, the left hand side is less than the right hand side at housing level H^o.

6.3 The conjectured equilibrium

On the basis of this understanding of the initial residents' optimal plan, we conjecture that equilibrium will have the following three features. First, the public good level will always be efficient. This means that for all states (g, b, H), $g'(g, b, H)$ will equal $(1-\delta)g^o(H'(g, b, H))$. Second, there will exist a *threshold wealth function*, which we denote $W^*(H)$, such that for all states (g, b, H) new construction will occur if the community's wealth exceeds this threshold.[24] Based on Proposition 3, we would expect this function to be increasing. Moreover, for housing levels in the interval $[H_0, H^s)$ it should exceed $\mathcal{W}(H)$ and for housing levels in the interval $[H^s, 1]$ it should equal $\mathcal{W}(H)$. Third, for states (g, b, H) such that the community's wealth is less than the threshold, the community will accumulate wealth. There will exist some increasing function $W_n(H)$ describing how much the community will accumulate. This function will have the property that $W_n(H)$ exceeds $W^*(H)$ for housing levels in the interval $[H_0, H^s)$, so that the wealth accumulation spurs development in the next period.

We also need to specify what happens in states in which new construction occurs. We conjecture that the community's policy choices cannot leave next period's residents with an incentive to build wealth. Were they to do so, the fall in house prices caused by the higher taxes would be anticipated and the market deterred from providing new construction in the current period. Leaving next period's residents with no incentive to build wealth will require that the new state (g', b', H') is such that W' is at least as big as $W^*(H')$. Since there would seem to be no gain to current residents from leaving more wealth than necessary, we conjecture that W' will equal $W^*(H')$.

Assuming all this to be true, the foundation of equilibrium will be the threshold wealth function $W^*(H)$. Once we have this, the other components of the equilibrium follow from it. To see this, let Ψ denote the set of all real valued, continuous functions W^* defined on the interval $[H_0, 1]$ with the properties that (i) W^* is increasing, differentiable, and exceeds $\mathcal{W}$ on the interval $[H_0, H^s)$, and (ii) W^* is equal to $\mathcal{W}$ on the interval $[H^s, 1]$. For any $W^* \in \Psi$, we can define a corresponding candidate equilibrium, which we denote $\mathcal{E}(W^*)$.

In the candidate equilibrium $\mathcal{E}(W^*)$, the public good and housing stock evolve according to

[24] It should be clear that this threshold wealth function $W^*(H)$ will be related to, but not exactly the same as, the threshold wealth function $W^*(H_t)$ associated with Proposition 3. While a different notation could have been employed to more clearly distinguish the two functions, for the remainder of the paper $W^*(H)$ will refer to the threshold wealth function associated with the equilibrium, so no confusion should arise.

the following rules:

$$(g'(g,b,H),H'(g,b,H)) = \begin{cases} ((1-\delta)g^o(H),H) & \text{if } W < W^*(H) \\ ((1-\delta)g^o(H_c(W)),H_c(W)) & \text{if } W \geq W^*(H) \end{cases}. \tag{25}$$

Here, W is equal to $cg-(1+\rho)b$ and the function $H_c(W)$ is defined implicitly as the solution to the system:

$$(1-H_c)\overline{\theta}+S(H_c)+\frac{W-\beta W^*(H_c)}{H_c}-C(1-\beta)=\underline{u} \;\;\&\;\; W^*(H_c)\geq W. \tag{26}$$

These rules embody the assumptions that the public good will be efficient and that new construction will not take place if wealth is below the threshold. When wealth is above the threshold, the new housing stock is $H_c(W)$. Intuitively, $H_c(W)$ is the housing level which, with current and future wealth levels W and $W^*(H_c)$, clears the market. It will be shown in the proof of the Theorem below that $H_c(W)$ is a well-defined and increasing function on the interval $[W^*(H_0),\mathcal{W}(1)]$.

The community's debt level evolves according to the rule

$$b'(g,b,H) = \begin{cases} \frac{c(1-\delta)g^o(H)-W_n(H)}{1+\rho} & \text{if } W < W^*(H) \\ \frac{c(1-\delta)g^o(H_c(W))-W^*(H_c(W))}{1+\rho} & \text{if } W \geq W^*(H) \end{cases}, \tag{27}$$

where the function $W_n(H)$ will be defined formally below once the value function has been introduced. Combining (25) and (27), we see these rules embody the assumption that when W exceeds $W^*(H)$ and new construction takes place, the community is left in the next period with a wealth level $W^*(H_c(W))$ exactly equal to the threshold level associated with its new housing stock. When W is smaller than $W^*(H)$, the community's wealth level in the next period is $W_n(H)$, so this represents the wealth level the community will build up to when wealth is below the threshold. The tax rule just follows from the budget constraint (1) and is given by

$$T(g,b,H)=\frac{(1+\rho)b+c\left(\frac{g'(g,b,H)}{1-\delta}-g\right)-b'(g,b,H)}{H'(g,b,H)}. \tag{28}$$

It remains to define the housing price rule and value function. In defining the former, we employ the notation

$$\mathcal{P}(H,W',W)=(1-H)\overline{\theta}+S(H)+\frac{W-\beta W'}{H}+\beta C-\underline{u}, \tag{29}$$

to denote the price at which housing demand would equal H if wealth levels this and next period were W and W', the housing price next period were C, and the community provides the efficient level of the public good. The housing price rule is then given by

$$P(g,b,H) = \begin{cases} \mathcal{P}(H, W_n(H), W) & \text{if } W < W^*(H) \\ C & \text{if } W \geq W^*(H) \end{cases} . \tag{30}$$

The fact that the housing price is described by the function $\mathcal{P}$ when there is no new construction reflects the fact that there will be new construction next period and hence a housing price of C.

The residents' value function is given by

$$V(g,b,H) = \begin{cases} V^*(W^*(H)) + \frac{W - W^*(H)}{H} & \text{if } W < W^*(H) \\ V^*(W) & \text{if } W \geq W^*(H) \end{cases} , \tag{31}$$

where the function $V^*(W)$ is defined recursively as

$$V^*(W) = (1-\mu)\left[C + \frac{u}{1-\beta}\right] + \mu\left[S\left(H_c(W)\right) + \frac{W - \beta W^*(H_c(W))}{H_c(W)} + \beta V^*(W^*(H_c(W)))\right]. \tag{32}$$

Finally, the function $W_n(H)$ is defined as

$$W_n(H) = \arg\max_{W'} \left\{ \begin{array}{c} (1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{u}{1-\beta}\right] + \mu\left[S\left(H\right) + \frac{W^*(H) - \beta W'}{H} + \beta V^*(W')\right] \\ s.t.\ \mathcal{P}(H, W', W^*(H)) \leq C \end{array} \right\}. \tag{33}$$

This maximization problem represents the problem faced by residents choosing next period's wealth under the constraint that there be no new construction this period. Notice that the price constraint implies that $W_n(H)$ must be at least as large as $W^*(H)$ which means that next period's housing price is C.

Our conjectured equilibrium is the candidate equilibrium $\mathcal{E}(W^*)$ associated with the threshold wealth function W^* that satisfies a particular property. This is that for all housing levels H, when the community has wealth $W^*(H)$, the residents are indifferent between choosing the equilibrium policies and the no new construction policies $(W_n(H), H)$.[25] Our next result verifies that, if the function $V^*(W)$ associated with this threshold wealth function is strictly concave, this "guess" is indeed an equilibrium.

Theorem *Suppose that Assumptions 1-3 are satisfied. Let $W^* \in \Psi$ and let $\mathcal{E}(W^*)$ be the associated candidate equilibrium. Suppose that (i) the function $V^*(W)$ defined in (32) is strictly concave on*

[25] By the no new construction policies $(W_n(H), H)$, we mean the policies that would keep housing constant, provide the public good efficiently, and result in a wealth level $W_n(H)$ next period.

$[W^*(H_0), \mathcal{W}(1)]$ *and (ii) for all* $H \in [H_0, 1)$

$$V^*(W^*(H)) = (1-\mu)\left[\mathcal{P}(H, W_n(H), W^*(H)) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W^*(H) - \beta W_n(H)}{H} + \beta V^*(W_n(H))\right]. \tag{34}$$

Then, $\mathcal{E}(W^*)$ *is an equilibrium.*

In light of this result, we will refer to a threshold wealth function that satisfies the conditions of the Theorem as an *equilibrium threshold wealth function.* To interpret the indifference condition (34), note that $V^*(W^*(H))$ (which is defined in (32)) represents the continuation payoff from the equilibrium policies when the initial state (g, b, H) is such that the community's wealth is $W^*(H)$. The expression on the right hand side represents the continuation payoff from building wealth up to $W_n(H)$, holding constant housing at H. The proof of the Theorem can be found in Appendix 2. The main task lies in establishing that the policy rules defined above are optimal for the residents in the sense of solving problem (17) when the function $V^*(W)$ is strictly concave and (34) holds.

6.4 Existence of equilibrium

The Theorem leaves open the question of whether there exists an equilibrium threshold wealth function. While we do not have an analytical proof that such a function must exist, we have been able to find equilibrium threshold wealth functions numerically for specific parameterizations of the model. Indeed, we have considered a vast number of such parameterizations and in almost all those satisfying our assumptions we have found an equilibrium threshold wealth function. Specifically, we have studied 20878 different parameterizations under which there exist an interval of initial housing levels H_0 that satisfy Assumptions 1-2. For over 95% of these, there exists an equilibrium threshold wealth function for every initial housing level in the interval. A detailed account of our numerical analysis of the model can be found in Appendix 4.

7 Equilibrium community development

This section describes how the community develops in the equilibrium found in the previous section.[26]

[26] For readers who have skipped Section 6, it will be useful to bear in mind the following definitions and concepts. First, W is the community's wealth, defined as $cg - (1+\rho)b$, and W_0 is its initial wealth. Second, $g^o(H)$ is the efficient level of the public good for a community of size H, as defined in (4). Third, $\mathcal{W}(H)$ is the wealth the community needs to attract a population of size H, if it provides the efficient level of the public good and finances it so as to keep its wealth constant. It is formally defined in (20). Fourth, $\mathcal{H}(W)$ is the inverse of $\mathcal{W}(H)$ and is therefore the population the community can attract with wealth level W if it provides the efficient level of the

7.1 High initial wealth

The easy case is when W_0 exceeds $\mathcal{W}(H^s)$. Then, the initial residents' optimal plan is time consistent and the equilibrium outcome is exactly as under this plan. Thus, in the initial period, residents invest in the public good and the market provides new construction. The increase in the stock of the public good is financed entirely with debt, which keeps the community's wealth constant. Thereafter, there is no further new construction.[27]

Proposition 5 *Suppose that Assumptions 1-3 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. If $W_0 \geq \mathcal{W}(H^s)$, then, in this equilibrium, in period 0, the community invests in $g^o(\mathcal{H}(W_0)) - g_0$ units of the public good and the market provides $\mathcal{H}(W_0) - H_0$ new houses. The community finances investment so as to keep its wealth constant, meaning that all but $c\delta g^o(\mathcal{H}(W_0))$ is financed with debt. Thereafter, the public good is maintained at $g^o(\mathcal{H}(W_0))$ and no more housing is provided. Taxes are set so that wealth remains at W_0. Throughout, the price of housing is C.*

7.2 Medium initial wealth

When W_0 is less than $\mathcal{W}(H^s)$, the initial residents' optimal plan is not time consistent, so the equilibrium outcome must differ from this plan. If W_0 is between $W^*(H_0)$ and $\mathcal{W}(H^s)$, the market provides new construction in the initial period, but less than the $\mathcal{H}(W_0) - H_0$ units provided under the path associated with the high initial wealth case. When the residents invest in the public good, they finance some of the investment with taxation and, as a result, the market provides less new construction. However, the next period, the community begins with a higher level of wealth. The same thing happens again in period 1: the residents finance some investment with taxes, which dampens new construction but increases the community's wealth a little further. This process keeps going indefinitely, with the community's housing and wealth levels gradually increasing. The size of the community converges to H^s asymptotically.

Proposition 6 *Suppose that Assumptions 1-3 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. If $W_0 \in [W^*(H_0), \mathcal{W}(H^s))$, then, in this equilibrium, the market provides new construction in every period and the housing stock con-*

public good and finances it so as to keep its wealth constant. Fifth, H^s is the smallest housing level H such that if the community has wealth $\mathcal{W}(H)$, residents have no incentive to build up wealth to attract more residents. It is formally defined in (24). Sixth, $W^*(H)$ is the threshold wealth function: when the community has housing stock H, new construction occurs if W exceeds $W^*(H)$ and new construction does not occur if W is less than $W^*(H)$. This threshold wealth function is the key object determined in equilibrium.

[27] The proofs of Propositions 5, 6, and 7 can be found in Appendix 3.

verges asymptotically to H^s. In all periods, the community provides the efficient level of the public good and finances some of the increase in stock with taxes. The community's wealth increases, converging asymptotically to $\mathcal{W}(H^s)$. Throughout, the price of housing is C.

7.3 Low initial wealth

If W_0 is smaller than $W^*(H_0)$, no new construction occurs in the initial period. Rather, the residents simply raise taxes to build up wealth. In the next period, new construction gets underway. Depending on the amount of wealth accumulation, there can either be a single period of new construction and the housing stock jumps to H^s, or there can be new construction in every period and gradual convergence to H^s.

Proposition 7 *Suppose that Assumptions 1-3 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. If $W_0 < W^*(H_0)$, then, in this equilibrium, the market provides no new construction in period 0 and the community raises taxes to build wealth to a level $W_n(H_0) \leq \mathcal{W}(H^s)$. If $W_n(H_0) < \mathcal{W}(H^s)$, the market provides new construction in every subsequent period, the housing stock converges asymptotically to H^s, and the community's wealth converges asymptotically to $\mathcal{W}(H^s)$. If $W_n(H_0) = \mathcal{W}(H^s)$, the market provides $H^s - H_0$ new houses in period 1, no new houses in subsequent periods, and wealth remains at $\mathcal{W}(H^s)$. In ether case, in all periods the community provides the efficient level of the public good. The price of housing is less than C in period 0 and C thereafter.*

7.4 The incentives underlying wealth accumulation

It is important to understand the incentives underlying the wealth accumulation that takes place in the medium and low initial wealth cases. The interesting point to note is that, when new construction occurs, the residents are not choosing to hold back current development to subsidize future development. On the contrary, they choose the largest amount of development they can and it is the equilibrium behavior of future residents that forces wealth accumulation.[28] To illustrate, consider the initial period in the medium wealth case. The equilibrium policy involves the residents increasing wealth and the market providing a level of new construction less than $\mathcal{H}(W_0) - H_0$. Why do not the residents avoid this wealth accumulation and let the market provide

[28] To clarify what we mean by the largest amount of development, suppose the state is such that the community has housing stock $H < H^s$ and wealth $W \geq W^*(H)$, so that equilibrium involves new construction. The equilibrium policies are such that the housing stock grows to $H_c(W)$ and wealth grows to $W^*(H_c(W))$. Consider some alternative policies (W', H') involving a higher level of new construction (i.e., H' exceeds $H_c(W)$). Then (W', H') is not a feasible policy choice given wealth level W.

$\mathcal{H}(W_0) - H_0$ units of new construction? The reason is that the market will not respond in this way. Market participants recognize that in response to this policy, next period's residents would build up the community's wealth by paying down some of its accumulated debt. The associated tax increase will mean that no development takes place and the price of housing will fall below C.[29] This fall in the future value of housing is anticipated by market participants in the initial period and dampens market demand.

It is in the initial period in the low wealth case, that the residents may choose to hold back development. In equilibrium, there is no new construction and the residents build the community's wealth to $W_n(H_0)$ by raising taxes, paving the way for development in period 1. As in the commitment case, this occurs even when development is feasible. In particular, if W_0 is smaller than but sufficiently close to $W^*(H_0)$, then there exists a housing level H' larger than H_0 such that if the residents provided a public good level $g^o(H')$ and financed it in such a way as to increase wealth to $W^*(H')$, the market would respond by providing $H' - H_0$ units of new construction.[30] These alternative policies offer a higher payoff in the initial period than the equilibrium policies as they involve both a higher housing price and more development. They also allow development to continue in the next period. However, they offer less payoff in the future, because the wealth level $W_n(H_0)$ exceeds $W^*(H')$ and permits more development in the next period.

7.5 Examples of accumulating wealth paths

To illustrate how the community develops with medium and low initial wealth, we compute our equilibrium for a particular example. The public good benefit function has the form: $B(z) = B_0 z^\sigma/\sigma$, $\sigma \in (0, 1)$ and the parameters have the following values:

Parameter	$\bar{\theta}$	β	μ	δ	σ	B_0	α	C	c	$\underline{u}$
Value	1	$\frac{1}{1.06}$	.96	.1	.5	.34	.6	20	1	0

The results are described in Figure 1.

The left hand panels illustrate a path in the medium initial wealth case. The top panel describes the equilibrium sequence of wealth and housing levels $\langle W_t, H_t \rangle_{t=0}^{\infty}$. Wealth is measured on the

[29] Specifically, the price of housing would equal $\mathcal{P}(\mathcal{H}(W_0), W_n(\mathcal{H}(W_0)), W_0)$ which is less than C (see proof of Theorem).

[30] We know that $\mathcal{P}(H_0, W^*(H_0), W^*(H_0))$ is larger than $\mathcal{P}(H_0, \mathcal{W}(H_0), \mathcal{W}(H_0))$ which is equal to C. Thus, for W_0 smaller than but sufficiently close to $W^*(H_0)$, it must be the case that $\mathcal{P}(H_0, W^*(H_0), W_0)$ is larger than C. Moreover, W_0 is less than $\mathcal{W}(H^s)$ which equals $W^*(H^s)$. It follows that $\mathcal{P}(H^s, W^*(H^s), W_0)$ is less than C. Thus, by continuity, there exist $H' \in (H_0, H^s)$ such that $\mathcal{P}(H', W^*(H'), W_0) = C$. Such a H' has the properties discussed in the text.

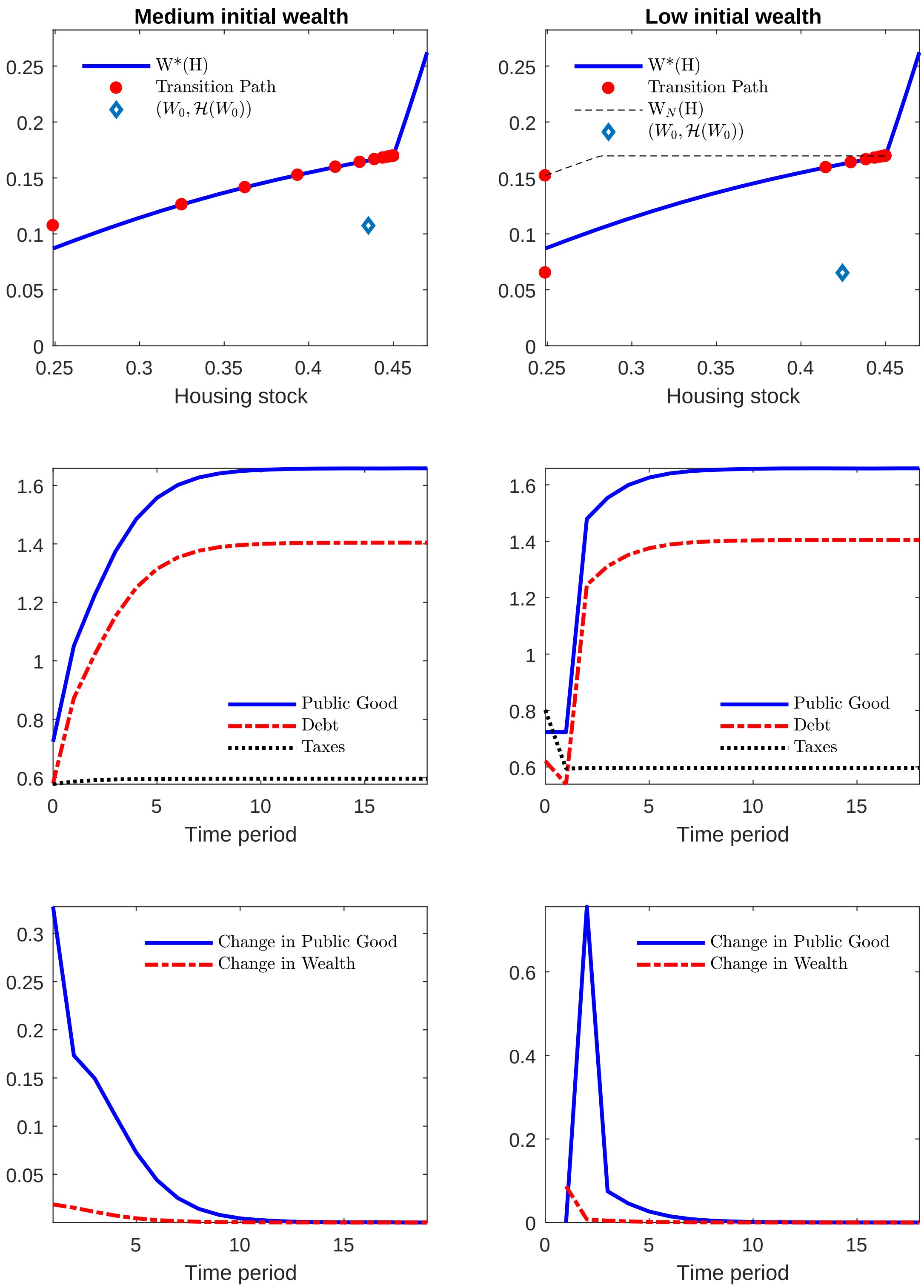

Figure 1: Accumulating wealth paths

vertical axis, and housing on the horizontal. The left most point is (W_0, H_0), the next one is (W_1, H_1), etc, etc. The line in this panel describes the equilibrium threshold wealth function $W^*(H)$, so the position of (W_0, H_0) implies that W_0 exceeds $W^*(H_0)$. Note that the threshold wealth function is kinked at the steady state housing level H^s. Reflecting the fact that W_t is equal to $W^*(H_t)$ for all periods beyond the initial period, (W_1, H_1) and the points that follow all lie on this line. In this example, $\mathcal{H}(W_0)$ is equal to 0.435, so that it takes the community six periods to exceed the housing level that would arise under the path associated with high initial wealth. The middle panel describes the evolution of the public good, taxes, and debt. The public good increases steadily as the population increases. Taxes increase, albeit very little, while debt grows. The growth in debt parallels the growth in the public good. The bottom panel illustrates how the community is building wealth. The upper line describes the increase in the community's public good stock, $g_t - g_{t-1}$, and the bottom line describes the increase in wealth, $W_t - W_{t-1}$. Given that the difference between the increase in public good stock and the increase in wealth is the increase in debt, the difference between the two lines illustrates how much of the increase in the public good is paid for with debt. While most of the increase is financed by debt, a proportion is financed by taxes which is what allows wealth to build. Given that taxes are essentially constant over the course of development, this tax finance is driven by the increase in tax revenues stemming from a larger tax base.

The right hand panels of Figure 1 illustrate a path in the low initial wealth case. The top panel again describes the equilibrium sequence of wealth and housing levels. The dashed line represents the function $W_n(H)$. In period 0, there is no new construction and the community accumulates wealth so that (W_1, H_1) equals to $(W_n(H_0), H_0)$. This sets the stage for a sharp increase in new construction in period 1, with the housing stock almost doubling between periods 1 and 2. Thereafter, new construction continues but at a miniscule pace. When compared with the left hand panel, it is clear that development is much more rapid, despite the fact that the community begins with less wealth. This rapid development is made possible by the sharp wealth accumulation in period 0. In this example, $\mathcal{H}(W_0)$ is equal to 0.424, so that it takes the community three periods to exceed the housing level that would arise if it did not accumulate wealth. The middle panel reveals that taxes fall sharply after period 0 and then increase very slowly. Debt falls in period 0 as tax revenues are used to reduce obligations. Debt then jumps up in period 1 as it is used to finance the large increase in the public good necessary to attract the new construction. After this, debt is rising slowly, reflecting the fact that little public good is added. Again, the growth in debt from period 1 onwards parallels the growth in the public good. The bottom panel illustrates the increase in wealth which occurs in period 0 with no corresponding increase in public good. After this, the increase in wealth is barely perceptible, which squares with the slow growth illustrated in the top panel.

8 Equilibrium vs optimal community development

A key motivation for the paper is to understand whether development proceeds efficiently and, if not, the nature of the distortions. Comparing Proposition 1 with Propositions 5, 6, and 7, yields the following conclusion.[31]

Proposition 8 *Suppose that Assumptions 1-3 are satisfied. Let $W^* \in \Psi$ be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Whatever W_0, some development occurs and, in all periods, the community provides the efficient level of the public good. If $W_0 \geq \mathcal{W}(H^s)$ the long-run size of the community in equilibrium will be larger (smaller) than optimal if $W_0 > (<) \mathcal{W}(H^o)$. If $W_0 < \mathcal{W}(H^s)$ the long-run size will be smaller than optimal, and, in addition, the equilibrium exhibits delay because development occurs after period* 0.

It is important to understand why these distortions in the path of development arise. Consider first the high initial wealth case (i.e., $W_0 \geq \mathcal{W}(H^s)$). In this case, there is no distortion in terms of timing, since all development occurs in the initial period. However, the community can be too small or too big depending on the relative sizes of W_0 and $\mathcal{W}(H^o)$. This reflects the two forces identified in the introduction: namely, potential residents and developers do not take into account the positive cost-sharing externality associated with their building houses but also have an incentive to free-ride on the community's assets. When W_0 exceeds $\mathcal{W}(H^o)$, the increase in public good surplus generated by the community's assets (which is $(1-\beta)W_0/H$) is more than sufficient to offset the cost-sharing externality (which is $HS'(H)$). When W_0 is less than $\mathcal{W}(H^o)$, the opposite is true. Using (7), (8), and (20), it is easy to show that

$$\mathcal{W}(H^o) = \frac{(1-\alpha)cg^o(H^o)(1-\beta(1-\delta))}{1-\beta}, \tag{35}$$

so that whether the community is too small or too large depends on the congestibility of the public good α. In the limit case α equal 1, $\mathcal{W}(H^o)$ equals 0 implying that, if the community begins with any positive wealth, W_0 will exceed $\mathcal{W}(H^o)$. At the other extreme of α equal 0, $\mathcal{W}(H^o)$ exceeds

[31] To understand this result and the discussion to follow, the reader will need to recall that H^o is the optimal housing level defined in (7) and to bear in mind that $H^s \in (H_0, H^o)$.

$cg^o(H^o)$. Thus, even if the community begins with the optimal level of public good for its initial population and no debt, W_0 will fall well short of $\mathcal{W}(H^o)$.

Why does the community not adjust its wealth to deal with these inefficiencies? When W_0 exceeds $\mathcal{W}(H^o)$, there is nothing the community can do to dissipate its wealth. The initial residents would like to ration entry into the community, but this is not possible given the available policies. If they tried to reduce the community's wealth by lowering taxes, this would not help because incoming households also benefit from these cuts. When W_0 is less than $\mathcal{W}(H^o)$, the initial residents could increase wealth. So too could the residents at the beginning of period 1. However, it is costly for them to do this in the short run and the future benefits are insufficient to offset these costs. This reflects the presence of an externality between different cohorts of residents. If by taking some costly action, current residents increase housing stocks in some future period t they will benefit all future potential residents from period t onward. Housing is infinitely durable, so that once additional housing has been built, the number of residents will be permanently increased. The missing market underlying this externality is that future potential residents cannot contribute to making the community more attractive to developers in the present. Borrowing does not resolve the problem, because the community needs to build wealth and this requires future potential residents to pay taxes today.

Now consider the medium initial wealth case (i.e., $W_0 \in [W^*(H_0), \mathcal{W}(H^s))$). The new distortion that arises is that development is delayed. To shed light on this, it is instructive to consider the policy choices in the initial period and analyze why residents prefer the equilibrium choices to those involving less delay. The equilibrium policies are such that the community's wealth and housing increase but to levels smaller than $(\mathcal{W}(H^s), H^s)$. Since the community's wealth and housing will eventually increase to $(\mathcal{W}(H^s), H^s)$, why do current residents not prefer to just jump directly to $(\mathcal{W}(H^s), H^s)$? The reason is that raising wealth to this level would require higher taxes and, given these taxes, the market would not provide $H^s - H_0$ new homes. This reflects the cost-sharing externality that potential residents and developers do not take into account. As noted earlier, residents are choosing the maximum level of development they can.

In the low initial wealth case (i.e., $W_0 < W^*(H_0)$), the equilibrium involves no new construction in period 0 and the community's wealth growing to $W_n(H_0)$. Development is delayed at least one period. Again, potential residents do not want to join the community given its current level of assets. If $W_n(H_0)$ is equal to $\mathcal{W}(H^s)$, then all development occurs in period 1 and there is no further delay. If $W_n(H_0)$ is less than $\mathcal{W}(H^s)$, then there is further delay as the housing stock gradually grows to H^s. The initial residents could instead choose to increase the community's wealth all the way to $\mathcal{W}(H^s)$, reducing delay to just one period. However, the tax burden of the increase in wealth must be bourne solely by the period 0 residents, whereas the benefits of the higher wealth are shared by the new residents. Thus, this will not always be an attractive strategy.[32]

Further insight into what is driving these distortions can be obtained by considering the limit as μ tends to 1 in which case residents know that they will remain in the community forever. From (24), we see that H^s tends to H^o. Moreover, it can be shown that the range of values of H_0 for which $W_n(H_0)$ equals $\mathcal{W}(H^s)$ vanishes. This means that if W_0 is below $\mathcal{W}(H^o)$, the community must gradually grow towards the optimal size H^o. It follows that, while population turnover is responsible for the fact that the community will be too small, it is not responsible for the fact that development is too slow. This reflects more fundamental forces.

It is natural to wonder if the distortions identified here could be eliminated if the residents had access to additional policy instruments. One such instrument is a subsidy or tax on new construction. A subsidy could in principle play a role in attracting new residents to the community and a tax could be used when necessary to deter entry. However, it is clear intuitively that such an instrument brings its own set of complications. This is because the level of subsidy or tax will impact the price of existing homes and residents care directly about this price. In Barseghyan and Coate (2017), we extend the model to allow for a new construction subsidy or tax. This is an interesting and challenging problem that relates to the classic literature on monopoly pricing of durable goods without commitment (see, for example, Bulow 1982 and Stokey 1981). Our preliminary results suggest that the long-run size of the community in equilibrium will always be smaller than optimal, and, in addition, the equilibrium always exhibits delay. While we have yet to complete a full welfare analysis, there seem to be a broad class of circumstances where allowing residents to subsidize or tax new construction will actually reduce welfare.

The distortions identified here not-withstanding, the equilibrium path of development is not without merit. First, development happens. Under our assumptions, no matter how small the community's initial wealth, some build up of wealth occurs and this creates some growth. Second, at all times, public good provision is efficient. This latter feature would seem to be a consequence of the community having access to debt. This provides two mechanisms by which to create wealth

[32] A sufficient condition for $W_n(H_0)$ to be less than $\mathcal{W}(H^s)$ is that H_0 is less than $(1-\beta)\,H^s/\mu\,(1-\mu\beta)$. Note that for sufficiently large μ, this condition is satisfied for any H_0 less than H^s.

- investing in the public good and reducing debt. While extending the analysis to see what would happen if the community could not borrow or save is another interesting and challenging problem, intuitively it seems likely that residents would have to overprovide the public good in order to make the community more attractive.

9 Conclusion

This paper has presented a model in which to consider community development. The model focuses on two key aspects of the development process: the expansion of housing and the build up of local public goods. Under the assumptions that investment decisions are made in each period by current residents and financed by debt or taxes, and that new housing is supplied by competitive developers, the paper has defined a notion of equilibrium development. It has found such an equilibrium and has analyzed what it looks like. This analysis yields a positive theory of how a community grows. This theory provides predictions concerning the time path of public investment, taxes, public debt, housing, and housing prices.

The theory predicts that a community's initial wealth (the value of its stock of public goods less its debt) determines how it develops. A community with high wealth develops rapidly. Residents finance the investment necessary to accomodate new residents entirely with debt, meaning that the community's wealth remains constant. A community with medium wealth develops gradually in the sense that new construction occurs each period and the housing stock converges asymptotically to a steady state. Residents finance the public investment necessary to accomodate new residents with a mix of debt and taxes, meaning that the community's wealth grows as it develops. This wealth accumulation is what spurs future development. A community with low wealth builds wealth before development begins. Wealth building is achieved by using taxes to reduce debt or increase the stock of public goods. After this period of wealth building, development takes place gradually or rapidly depending on the amount of wealth accumulated.

These predictions highlight the potential importance of a community's public assets in determining its development path. They also focus attention on the improvement of public assets as a development strategy. While the public and urban economics literatures do not appear to have analyzed this strategy, it makes sense in dynamic settings where new residents confer a positive externality on existing residents and new construction is determined by the market. Nonetheless, since improving public assets imposes costs on existing residents and may dampen short-run development, it is by no means obvious that it could arise in a political economy setting where policies are chosen each period by current residents. The theory explains how wealth accumulation arises in such a setting and the incentives that shape it.

The theory provides a number of interesting normative lessons. First, residents will always provide the efficient level of public good for their community. This reflects the fact that what matters for development is the community's wealth and any given target level of wealth can be achieved by varying debt, so there is no need to distort the public good. Second, when a community has good public assets, it is a possible for it to experience too much development. This reflects the fact that potential residents and developers free ride on a community's public assets. Third, when a community's public assets are such that improvement is necessary to achieve optimal size, the community will end up undersized. This reflects the fact that, while the costs of improving public assets are bourne by current residents, some of the benefits are obtained by future residents. Accordingly, current residents do not appropriate the full fruits of their sacrifice. Fourth, when wealth accumulation is necessary for a community to achieve optimal size, development may be delayed. The gradual wealth accumulation that emerges in equilibrium reflects the fact that, because the community is growing in size, future residents have a greater incentive to build wealth than current residents.

There are many ways the theory could usefully be developed to shed further light on the dynamics of community development. Working with the same underlying economic model but changing the policy space, one could explore how development changes when residents have access to different public finance instruments.[33] We have already discussed two important extensions of this form: allowing the community to impose a subsidy or tax on new construction, and assuming that the community cannot borrow or save. One could also change or enrichen the underlying economic model and see how this impacts development. For example, what would happen if the optimized public good surplus function were hump-shaped rather than increasing, so that the externality associated with household entry could become negative?[34] In addition, what

[33] Brueckner (1997) studies the impact of different methods of financing infrastructure on the pattern of urban growth in a monocentric city model in which growth is driven by exogenous changes in the income available from working in the city. Brueckner's model is very different from that considered here in that it is a spatial model in which all residents are renters, all taxes are paid by the owners of developed land, and all policies are exogenous. Brueckner considers three different financing methods: uniform taxation of the owners of developed land, taxes on newly developed land, and debt (which effectively taxes future development). Crucially, these three methods provide different incentives for landowners to develop their land.

[34] Following Tiebout (1956), it is common in the literature to assume that communities face a U-shaped average cost of supplying any given level of public services. This could generate a hump-shaped optimized public good surplus function. To explore such a possibility formally, one could assume that the level of public services experienced by residents was some function $f(g'/(1-\delta), H)$ and that residents obtained benefits from public services $B(f(g'/(1-\delta), H))$. By appropriate choice of the function f, a hump-shaped $S(H)$ function could be generated.

would happen if there was an upward sloping supply curve of land, say because land had variable agricultural productivity? This possibility would be particularly interesting as it would require modelling the political conflict between landowners/developers and residents. In this vein, it might also be interesting to assume that the community is initially empty and that all land is owned by landowners/developers. One could then try to model the transition of political control from developers to residents.[35]

Finally, the theory could be adapted to study the dynamic development of clubs (country clubs, golf clubs, social clubs, hobby clubs, community associations, etc). Essentially, this would involve studying this model with no housing. In this conception, a club would be characterized by its membership size, its stock of public good, and its debt. In each period, the club invests in the public good and finances this either by a tax on members or a debt issue. Club decisions are made collectively by current members. The only difference between this and our model, is that individuals do not have to buy a house to join the club. They just have to pay the tax (i.e., membership fee) to benefit. Any existing member can avoid the tax by just leaving the club. In particular, they do not have to sell their house. In a sense, this set-up is simpler than the model considered here, because there is no housing market. On the other hand, the lack of durable housing means there is nothing to anchor club size: in particular, it could shrink.

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[35] See Chapter 7 of Fischel (2015) for discussion of this process.

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10 Appendix 1: Proof of Propositions 2 and 3

Since Proposition 3 is a generalization of Proposition 2, it suffices to just prove Proposition 3. We begin with a statement of the problem faced by the period t residents.

10.1 Statement of the problem

The period t residents' problem is

$$\max_{\{g_{\tau+1},b_{\tau+1},T_\tau,H_{\tau+1},P_\tau\}_{\tau=t}^{\infty}} \left\{ \begin{array}{c} \sum_{\tau=t}^{\infty} (\mu\beta)^{\tau-t} \left\{(1-\mu)\left[P_\tau + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau\right]\right\} \\ s.t.\ (1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau\right) = b_{\tau+1} + H_{\tau+1}T_\tau \\ g_{\tau+1} \geq 0 \ \&\ H_{\tau+1} \geq H_\tau \\ P_\tau = (1-H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau + \beta P_{\tau+1} - \underline{u} \\ P_\tau \leq C\ (=\text{if } H_{\tau+1} > H_\tau) \end{array} \right\}. \tag{36}$$

The first constraint is the budget constraint. The second and third constraints are the feasibility constraints. The non-negativity constraint on $g_{\tau+1}$ will henceforth be ignored, as it will not be binding. The fourth and fifth constraints are the market equilibrium constraints. The residents also need to respect the transversality condition that $\lim_{\tau\to\infty}\beta^\tau b_\tau = 0$, to prevent them operating a Ponzi scheme. Finally, the residents face initial conditions (g_t, b_t, H_t). We assume that $H_t \in [H_0, 1]$ and that $W_t \in [W_0, \mathcal{W}(1)]$, where (W_0, H_0) satisfy Assumptions 1-3.

10.2 Solving the problem

Problem (36) involves too many constraints to tackle head on by forming a Lagrangian. Rather, we must approach it through a process of simplification. Our first result concerns the objective function.

Fact A.1.1. *Suppose that the sequence of policies $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies in each period τ the market equilibrium condition*

$$P_\tau = (1-H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau + \beta P_{\tau+1} - \underline{u} \tag{37}$$

and the price constraint $P_t \leq C$. Then, the period t residents' objective function satisfies

$$\begin{aligned} &\sum_{\tau=t}^{\infty} (\mu\beta)^{\tau-t} \left\{(1-\mu)\left[P_\tau + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau\right]\right\} \\ &\quad = P_t + \mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1) + \frac{\underline{u}}{1-\beta}. \end{aligned} \tag{38}$$

Proof of Fact A.1.1. From the market equilibrium condition, we have that

$$B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-T_{\tau}=P_{\tau}+(H_{\tau+1}-1)\overline{\theta}-\beta P_{\tau+1}+\underline{u}.$$

Thus, for any period $\tau\geq t$, we have that

$$\begin{aligned}&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g_{z+1}/(1-\delta)}{(H_{z+1})^{\alpha}}\right)-T_z\right]\right\}\\ =\ &\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{u}{1-\beta}\right]+\mu\left[P_z+(H_{z+1}-1)\overline{\theta}-\beta P_{z+1}+\underline{u}\right]\right\}.\end{aligned}$$

Expanding the right hand side, we have that

$$\begin{aligned}&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{u}{1-\beta}\right]+\mu\left[P_z+(H_{z+1}-1)\overline{\theta}-\beta P_{z+1}+\underline{u}\right]\right\}\\ =\ &(1-\mu)\left[P_t+\frac{u}{1-\beta}\right]+\mu\left[P_t+(H_{t+1}-1)\overline{\theta}-\beta P_{t+1}+\underline{u}\right]\\ &+\mu\beta\left\{(1-\mu)\left[P_{t+1}+\frac{u}{1-\beta}\right]+\mu\left[P_{t+1}+(H_{t+2}-1)\overline{\theta}-\beta P_{t+2}+\underline{u}\right]\right\}\\ &+(\mu\beta)^2\left\{(1-\mu)\left[P_{t+2}+\frac{u}{1-\beta}\right]+\mu\left[P_{t+2}+(H_{t+3}-1)\overline{\theta}-\beta P_{t+3}+\underline{u}\right]\right\}\\ &+.....+(\mu\beta)^{\tau-t}\left\{(1-\mu)\left[P_{\tau}+\frac{u}{1-\beta}\right]+\mu\left[P_{\tau}+(H_{\tau+1}-1)\overline{\theta}-\beta P_{\tau+1}+\underline{u}\right]\right\}.\end{aligned}$$

Note that the P_{t+1} term in the first line cancels with that in the second line. Similarly, the P_{t+2} term in the second line cancels with the P_{t+2} term in the third line, etc, etc. Thus, we have

$$\begin{aligned}&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{u}{1-\beta}\right]+\mu\left[P_z+(H_{z+1}-1)\overline{\theta}-\beta P_{z+1}+\underline{u}\right]\right\}\\ =\ &P_t+\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\frac{u}{1-\beta}+\mu\left[(H_{z+1}-1)\overline{\theta}+\underline{u}\right]\right\}-(\mu\beta)^{\tau+1-t}P_{\tau+1}.\end{aligned}$$

Given that $P_\tau\leq C$ for all τ, we have that $\lim_{\tau\to\infty}(\mu\beta)^{\tau+1-t}P_{\tau+1}=0$. Thus,

$$\begin{aligned}&\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}\left\{(1-\mu)\left[P_{\tau}+\frac{u}{1-\beta}\right]+\mu\left[P_{\tau}+(H_{\tau+1}-1)\overline{\theta}-\beta P_{\tau+1}+\underline{u}\right]\right\}\\ =\ &P_t+\mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1)+\frac{u}{1-\beta}.\end{aligned}$$

■

This result reveals that housing prices have a direct impact on the objective function only in the initial period and that, all else equal, the residents prefer to have future housing stocks as high as possible.

Our second result concerns the inter-temporal implications of the sequence of budget constraints.

Fact A.1.2. *Suppose that the sequence of policies* $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}\}_{\tau=t}^{\infty}$ *satisfies in each period* τ *the budget constraint*

$$(1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau\right) = b_{\tau+1} + H_{\tau+1}T_\tau \tag{39}$$

and that $\lim_{\tau\to\infty} \beta^\tau g_\tau = \lim_{\tau\to\infty} \beta^\tau b_\tau = 0$*. Then,* $\{g_{\tau+1}, T_\tau, H_{\tau+1}\}_{\tau=t}^{\infty}$ *satisfies the intertemporal budget constraint*

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta} - \beta g_{\tau+1}\right) - H_{\tau+1}T_\tau\right] = W_t. \tag{40}$$

Proof of Fact A.1.2. Suppose that the sequence of policies $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}\}_{\tau=t}^{\infty}$ satisfies in each period τ the budget constraint (39). From the period t budget constraint, we have that

$$c\frac{g_{t+1}}{1-\delta} - b_{t+1} - H_{t+1}T_t = W_t,$$

and from the period $t+1$ budget constraint, we have that

$$b_{t+1} = \beta\left(b_{t+2} + H_{t+2}T_{t+1} - c\left(\frac{g_{t+2}}{1-\delta} - g_{t+1}\right)\right).$$

Substituting the latter into the former yields

$$c\left(\frac{g_{t+1}}{1-\delta} - \beta g_{t+1}\right) + \beta c\frac{g_{t+2}}{1-\delta} - \beta b_{t+2} - H_{t+1}T_t - \beta H_{t+2}T_{t+1} = W_t.$$

Similarly, the period $t+2$ budget constraint tells us that

$$b_{t+2} = \beta\left(b_{t+3} + H_{t+3}T_{t+2} - c\left(\frac{g_{t+3}}{1-\delta} - g_{t+2}\right)\right).$$

Substituting this into the period t budget constraint yields

$$c\left(\frac{g_{t+1}}{1-\delta} - \beta g_{t+1}\right) + \beta\left(c\frac{g_{t+2}}{1-\delta} - \beta g_{t+2}\right) + \beta^2 c\frac{g_{t+3}}{1-\delta} - \beta^2 b_{t+3} - H_{t+1}T_t - \beta H_{t+2}T_{t+1} - \beta^2 H_{t+3}T_{t+2} = W_t.$$

Iterating this logic, we find that for all periods $\tau \geq t$

$$\sum_{z=t}^{\tau} \beta^{z-t}\left[c\left(\frac{g_{z+1}}{1-\delta} - \beta g_{z+1}\right) - H_{z+1}T_z\right] + \beta^{\tau-t}\left(cg_{\tau+1} - b_{\tau+1}\right) = W_t$$

Since $\lim_{\tau\to\infty} \beta^{\tau-t}\left(cg_{\tau+1} - b_{\tau+1}\right) = 0$, this implies that

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta} - \beta g_{\tau+1}\right) - H_{\tau+1}T_\tau\right] = W_t,$$

which is (40). ■

The assumed properties of the public good benefit function B imply that the public good level will be bounded above and hence that $\lim_{\tau\to\infty} \beta^\tau g_\tau = 0$. Moreover, the transversality condition requires that $\lim_{\tau \to \infty} \beta^\tau b_\tau = 0$. Thus, Fact A.1.2 suggests replacing the sequence of budget constraints (39) in the period t residents' problem with the single intertemporal budget constraint (40). Indeed, this is a standard procedure in models of optimal policy in which decision-makers face a sequence of budget constraints and have access to bonds. However, in our problem, we also have the market equilibrium constraints to worry about and these depend on the time path of taxes and hence local government debt. Specifically, while constraint (40) is independent of the time path of debt and just depends on the present value of taxes, the market equilibrium conditions (37) do depend on the time path of taxes. Thus, we need to verify that replacing the sequence of budget constraints with the single intertemporal budget constraint would not create problems in satisfying the market equilibrium conditions (37). This is confirmed by our next result.

Fact A.1.3. *Suppose that the sequence of policies $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies in each period τ the market equilibrium condition (37) and the budget constraint (39) and that* $\lim_{\tau \to \infty} \beta^\tau g_\tau = \lim_{\tau \to \infty} \beta^\tau b_\tau = 0$. *Then, $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies the constraints that*

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[c\left(\frac{g_{\tau+1}}{1-\delta} - \beta g_{\tau+1} \right) - H_{\tau+1} \left((1-H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}} \right) - P_\tau + \beta P_{\tau+1} - \underline{u} \right) \right] = W_t, \tag{41}$$

and that, for all periods $\tau \geq t$,

$$P_\tau = (1-H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}} \right) - \left(\frac{(1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau \right) - b_{\tau+1}}{H_{\tau+1}} \right) + \beta P_{\tau+1} - \underline{u}. \tag{42}$$

Conversely, if $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies the constraints (41) and, for all $\tau \geq t$, (42), then there exists $\{T_\tau\}_{\tau=t}^{\infty}$ such that $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies for all τ the market equilibrium condition (37) and the budget constraint (39).

Proof of Fact A.1.3. Let $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ be a sequence of policies satisfying in each period τ the market equilibrium condition (37), the budget constraint (39) and the requirement that $\lim_{\tau \to \infty} \beta^\tau g_\tau = \lim_{\tau \to \infty} \beta^\tau b_\tau = 0$. Then we know from Fact A.1.2 that

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[c\left(\frac{g_{\tau+1}}{1-\delta} - \beta g_{\tau+1} \right) - H_{\tau+1} T_\tau \right] = W_t. \tag{43}$$

Using (37) to solve for T_τ and substituting this into (43) yields (41). That (42) holds follows immediately from (37) after solving (39) for T_τ and substituting in for T_τ.

For the converse, let $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfy the constraints (41) and, for all $\tau \geq t$, (42). For all τ, let

$$T_\tau = (1 - H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^\alpha}\right) - P_\tau + \beta P_{\tau+1} - \underline{u}. \tag{44}$$

Simply rearranging this equation reveals that $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies (37) for all τ. Solving (44) for P_τ and using (42) reveals that, for all periods τ,

$$0 = T_\tau - \left(\frac{(1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau\right) - b_{\tau+1}}{H_{\tau+1}}\right).$$

Rearranging this equation yields (39). ■

Note that equation (41) is obtained from the intertemporal budget constraint by substituting in the expression for the tax rate implied by the market equilibrium condition. Equation (42) is obtained from the market equilibrium condition by substituting in the expression for the tax rate that is implied by the budget constraint. Thus, both equations reflect both the market equilibrium conditions and the budget constraint. Fact A.1.3 tells us that we can replace the per-period budget constraint and the market equilibrium condition with equations (41) and (42). It also allows us to eliminate T_t from the set of choice variables. Thus, we can recast the initial residents' problem as follows:

$$\max_{\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1} - 1) + \frac{\underline{u}}{1-\beta} \\ s.t.\ H_{\tau+1} \geq H_t,\ (41),\ (42),\ P_\tau \leq C\ (= \text{if } H_{\tau+1} > H_\tau) \end{array} \right\} \tag{45}$$

Our next result shows that there is no loss of generality in requiring the period t residents choose policies so that the price of housing is equal to C in all periods except period t.

Fact A.1.4. *Let $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ be a sequence of policies satisfying the constraints (41) and, for all τ, (42). Then, there exists $\{\widetilde{b}_\tau, \widetilde{P}_\tau\}_{\tau=t}^{\infty}$ such that $\widetilde{P}_\tau = C$ for all $\tau \geq t+1$, $\widetilde{P}_t = P_t$, and $\widetilde{b}_t = b_t$, with the property that $\{g_{\tau+1}, \widetilde{b}_{\tau+1}, H_{\tau+1}, \widetilde{P}_\tau\}_{\tau=t}^{\infty}$ satisfies the constraints (41) and, for all τ, (42).*

Proof of Fact A.1.4. Let $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ be a sequence of policies satisfying the constraints (41) and, for all τ, (42). Let $\{\widetilde{P}_\tau\}_{\tau=t}^{\infty}$ be such that $\widetilde{P}_\tau = C$ for all $\tau \geq t+1$ and $\widetilde{P}_t = P_t$. Then we claim that $\{g_{\tau+1}, H_{\tau+1}, \widetilde{P}_\tau\}_{\tau=t}^{\infty}$ satisfies the intertemporal budget constraint

(41). To prove this, it suffices to show that

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[c\left(\tfrac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\tfrac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-P_{\tau}+\beta P_{\tau+1}-\underline{u}\right)\right]=$$
$$\left[c\left(\tfrac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\tfrac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_{t}+\beta C-\underline{u}\right)\right]+$$
$$\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\left[c\left(\tfrac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\tfrac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right)\right].$$

For this, we need to show that

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}H_{\tau+1}\left(P_{\tau}-\beta P_{\tau+1}\right)=H_{t+1}\left(P_{t}-\beta C\right)+\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}H_{\tau+1}C\left(1-\beta\right).$$

We have

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}H_{\tau+1}\left(P_{\tau}-\beta P_{\tau+1}\right)=H_{t+1}\left(P_{t}-\beta P_{t+1}\right)+\beta H_{t+2}\left(P_{t+1}-\beta P_{t+2}\right)+\beta^{2}H_{t+3}\left(P_{t+2}-\beta P_{t+3}\right)+...$$

Let $z\geq t+1$ be the first time period in which $P_z<C$. Then, $H_z=H_{z+1}$. Thus,

$$\begin{aligned}
&\beta^{z-1-t}H_{z}\left(C-\beta P_{z}\right)+\beta^{z-t}H_{z+1}\left(P_{z}-\beta P_{z+1}\right)\\
=\;&\beta^{z-1-t}H_{z}\left[C-\beta P_{z}+\beta\left(P_{z}-\beta P_{z+1}\right)\right]\\
=\;&\beta^{z-1-t}H_{z}\left(C-\beta C\right)+\beta^{z-t}H_{z+1}\left(C-\beta P_{z+1}\right).
\end{aligned}$$

Continuing this argument yields the result.

Now define the sequence of bond levels $\{\widetilde{b}_{\tau+1}\}_{\tau=t}^{\infty}$ inductively as follows:

$$\widetilde{b}_{t+1}=c\frac{g_{t+1}}{1-\delta}-W_{t}-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_{t}+\beta C-\underline{u}\right)$$

and

$$\widetilde{b}_{\tau}=(1+\rho)\widetilde{b}_{\tau-1}+c\left(\frac{g_{\tau}}{1-\delta}-g_{\tau-1}\right)-H_{\tau}\left((1-H_{\tau})\overline{\theta}+B\left(\frac{g_{\tau}/(1-\delta)}{(H_{\tau})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right).$$

Then we have that

$$P_{t}=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_{t}+c\left(\frac{g_{t+1}}{1-\delta}-g_{t}\right)-\widetilde{b}_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u}$$

and, for all $\tau\geq t+1$

$$C=(1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-\left(\frac{(1+\rho)\widetilde{b}_{\tau}+c\left(\frac{g_{\tau+1}}{1-\delta}-g_{\tau}\right)-\widetilde{b}_{\tau+1}}{H_{\tau+1}}\right)+\beta C-\underline{u}.$$

■

Fact A.1.4 allows us to write the intertemporal budget constraint (41) as

$$c\left(\frac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right)+$$
$$\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right)\right]=W_t. \tag{46}$$

The market equilibrium constraints can be written as

$$P_t=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_t+c\left(\frac{g_{t+1}}{1-\delta}-g_t\right)-b_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u}, \tag{47}$$

and for all $\tau\geq t+1$

$$C=(1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_\tau+c\left(\frac{g_{\tau+1}}{1-\delta}-g_\tau\right)-b_{\tau+1}}{H_{\tau+1}}\right)+\beta C-\underline{u}. \tag{48}$$

Finally, the constraint that $P_\tau\leq C$ (with equality if $H_{\tau+1}>H_\tau$) need only be imposed for period t.

Next observe that given a choice of period $t+1$ debt, b_{t+1}, and a sequence of public good and housing levels $\{g_{\tau+1},H_{\tau+1}\}_{\tau=t}^{\infty}$, the market equilibrium constraints (48) pin down the sequence of debt levels $\{b_{\tau+1}\}_{\tau=t+1}^{\infty}$. Thus, these constraints can be eliminated and the debt levels $\{b_{\tau+1}\}_{\tau=t+1}^{\infty}$ can be removed from the set of choice variables. This allows us to write the period t residents' problem as:

$$\max_{\{P_t,b_{t+1},\{g_{\tau+1},H_{\tau+1}\}_{\tau=t}^{\infty}\}} P_t+\mu\overline{\theta}\textstyle\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1)+\frac{\underline{u}}{1-\beta}$$
$$s.t.\ H_{\tau+1}\geq H_\tau$$
$$c\left(\frac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right)+$$
$$\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right)\right]=W_t \tag{49}$$
$$P_t=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_t+c\left(\frac{g_{t+1}}{1-\delta}-g_t\right)-b_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u}$$
$$P_t\leq C\ (=\text{if } H_{t+1}>H_t)$$

Our next result uses this formulation to tie down the public good levels.

Fact A.1.5. *In the period t residents' optimal plan, for all periods $\tau\geq t$*

$$g_{\tau+1}=(1-\delta)g^{o}\left(H_{\tau+1}\right).$$

Proof of Fact A.1.5. Inspecting problem (49), it is clear that for all periods $\tau \geq t+1$, $g_{\tau+1}$ must equal $(1-\delta)g^o(H_{\tau+1})$. The only place $g_{\tau+1}$ enters the problem is in the intertemporal budget constraint and setting $g_{\tau+1}$ equal to the level $(1-\delta)g^o(H_{\tau+1})$ maximally relaxes this constraint. Given this, we can eliminate $\{g_{\tau+1}\}_{\tau=t+1}^{\infty}$ from the choice variables and use the definition of $\mathcal{W}(H_{\tau+1})$ to write the intertemporal budget constraint as:

$$c\left(\tfrac{g_{t+1}}{1-\delta} - \beta g_{t+1}\right) - H_{t+1}\left((1-H_{t+1})\overline{\theta} + B\left(\tfrac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - P_t + \beta C - \underline{u}\right) \\ + \sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\mathcal{W}(H_{\tau+1})(1-\beta) = W_t.$$

In addition, recalling that $W_\tau = cg_\tau - (1+\rho)b_\tau$, we can write the period t equilibrium market equilibrium constraint as

$$P_t = (1-H_{t+1})\overline{\theta} + B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - \left(\frac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}.$$

Substituting this into the intertemporal budget constraint, we get

$$\sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\mathcal{W}(H_{\tau+1})(1-\beta) = \beta W_{t+1}.$$

Thus, we can rewrite problem (49) as

$$\begin{gathered} \max_{\{P_t, W_{t+1}, g_{t+1}, \{H_{\tau+1}\}_{\tau=t}^{\infty}\}} P_t + \mu\overline{\theta}\textstyle\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1) + \frac{\underline{u}}{1-\beta} \\ s.t.\ H_{\tau+1} \geq H_\tau \\ \sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\mathcal{W}(H_{\tau+1})(1-\beta) = \beta W_{t+1} \\ P_t = (1-H_{t+1})\overline{\theta} + B\left(\tfrac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - \left(\tfrac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right) + \left(\tfrac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u} \\ P_t \leq C\ (= \text{if } H_{t+1} > H_t) \end{gathered} \tag{50}$$

We can now prove that g_{t+1} must equal $(1-\delta)g^o(H_{t+1})$. It is clear that this must be true if $P_t < C$, since then the optimal g_{t+1} must maximize P_t and hence

$$B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - \left(\frac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right).$$

In this case, we have that

$$P_t = (1-H_{t+1})\overline{\theta} + S(H_{t+1}) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}.$$

Suppose then that $P_t = C$ and that, contrary to our claim, g_{t+1} is not equal to $(1-\delta)g^o(H_{t+1})$. Then, we know that

$$\begin{aligned} C &= (1-H_{t+1})\overline{\theta} + B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - \left(\frac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u} \\ &< (1-H_{t+1})\overline{\theta} + S(H_{t+1}) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}. \end{aligned}$$

As an alternative to the policies (W_{t+1}, g_{t+1}), consider the policies $\left(W'_{t+1}, (1-\delta)g^o(H_{t+1})\right)$ where

$$C = (1-H_{t+1})\overline{\theta} + S(H_{t+1}) + \left(\frac{W_t - \beta W'_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}.$$

These policies do not change the price P_t. However, since $W'_{t+1} > W_{t+1}$, they relax the intertemporal budget constraint in the sense that

$$\sum_{\tau=t+1}^{\infty} \beta^{\tau-t}\mathcal{W}(H_{\tau+1})(1-\beta) = \beta W_{t+1} < \beta W'_{t+1}.$$

This permits the choice of a preferred sequence of housing levels $\{H'_{\tau+1}\}_{\tau=t}^{\infty}$, which is a contradiction. ■

Fact A.1.5 allows us to eliminate $\{g_{\tau+1}\}_{\tau=t}^{\infty}$ from the choice variables. Moreover, substituting in the optimal public good levels and using the definitions of $\mathcal{W}(H_{\tau+1})$ and W_τ, we can write the period t residents' problem as

$$\max_{\{P_t, W_{t+1}, \{H_{\tau+1}\}_{\tau=t}^{\infty}\}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1) + \frac{\underline{u}}{1-\beta} \\ s.t.\ H_{\tau+1} \geq H_\tau \\ \frac{H_{t+1}(P_t - C)}{1-\beta} + \sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathcal{W}(H_{\tau+1}) = \frac{W_t}{1-\beta} \\ P_t = C - \frac{(1-\beta)\mathcal{W}(H_{t+1}) - (W_t - \beta W_{t+1})}{H_{t+1}} \\ P_t \leq C\ (= \text{if } H_{t+1} > H_t) \end{array} \right\} \tag{51}$$

This formulation is particularly insightful as discussed in the text. Using it, we are able to show that all new construction takes place in periods t and $t+1$.

Fact A.1.6. *In the initial residents' optimal plan, for all $\tau \geq t+3$*

$$H_\tau = H_{t+2}.$$

Proof of Fact A.1.6. Suppose the Fact is not true. Let $\tau \geq t+3$ be the first period which violates the claim; that is,

$$H_\tau > H_{\tau-1} = ... = H_{t+2}.$$

Let λ be the multiplier on the intertemporal budget constraint in problem (51). Given that we can raise $H_{\tau-1}$ marginally without violating any of the constraints, we must have that benefits from so doing, which are $\mu\overline{\theta}\left(\mu\beta\right)^{\tau-2-t}$, are no greater than the costs, which are $\lambda\beta^{\tau-2-t}\mathcal{W}'(H_{\tau-1})$. This implies that

$$\mu\overline{\theta}\mu^{\tau-2-t} \leq \lambda\mathcal{W}'(H_{\tau-1}).$$

Given that we can lower H_τ marginally without violating any of the constraints, we must have that the benefits from so doing, which are $\lambda\beta^{\tau-1-t}\mathcal{W}'(H_\tau)$, are less than the costs, which are $\mu\overline{\theta}\left(\mu\beta\right)^{\tau-1-t}$. This implies that

$$\lambda\mathcal{W}'(H_\tau) \leq \mu\overline{\theta}\mu^{\tau-1-t}.$$

Combining these two inequalities we have that $\mathcal{W}'(H_\tau) < \mathcal{W}'(H_{\tau-1})$, which contradicts the fact that $\mathcal{W}(H)$ is convex. ■

Fact A.1.6 allows us to eliminate the housing levels $\{H_{\tau+1}\}_{\tau=t+2}^{\infty}$ from the choice variables and write the period t residents' problem as

$$\max_{\{P_t, W_{t+1}, H_{\tau+1}, H_{t+2}\}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta}\left[H_{t+1} - 1 + \frac{\mu\beta}{1-\mu\beta}(H_{t+2}-1)\right] + \frac{u}{1-\beta} \\ s.t.\ H_{t+2} \geq H_{t+1} \geq H_t \\ \frac{H_{t+1}(P_t - C)}{1-\beta} + \mathcal{W}(H_{t+1}) + \frac{\beta}{1-\beta}\mathcal{W}(H_{t+2}) = \frac{W_t}{1-\beta} \\ P_t = C - \frac{(1-\beta)\mathcal{W}(H_{t+1}) - (W_t - \beta W_{t+1})}{H_{t+1}} \\ P_t \leq C\ (= \text{if } H_{t+1} > H_t) \end{array} \right\}. \tag{52}$$

Moreover, plugging the market equilibrium constraint into the intertemporal budget constraint reveals that

$$\frac{W_t - \beta W_{t+1}}{1-\beta} + \frac{\beta}{1-\beta}\mathcal{W}(H_{t+2}) = \frac{W_t}{1-\beta},$$

which immediately implies that $H_{t+2} = \mathcal{H}(W_{t+1})$. The period t residents' problem then reduces to

$$\max_{\{W_{t+1}, H_{\tau+1}\}} \left\{ \begin{array}{c} C - \frac{(1-\beta)\mathcal{W}(H_{t+1}) - (W_t - \beta W_{t+1})}{H_{t+1}} + \mu\overline{\theta}\left[H_{t+1} - 1 + \frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_{t+1}) - 1)\right] + \frac{u}{1-\beta} \\ s.t.\ \mathcal{H}(W_{t+1}) \geq H_{t+1} \geq H_t \\ W_t - \beta W_{t+1} \leq (1-\beta)\mathcal{W}(H_{t+1})\ (= \text{if } H_{t+1} > H_t) \end{array} \right\} \tag{53}$$

This problem involves just a choice of two variables - W_{t+1} and $H_{\tau+1}$ - how much wealth to accumulate in period t and how much new construction to undertake.

Analyzing this problem, reveals that:

Fact A.1.7. *In the period t residents' optimal plan, there exist wealth levels $W^*(H_t)$ and $W_n(H_t)$, satisfying $\mathcal{W}(H_t) \leq W^*(H_t) \leq W_n(H_t)$, such that*

$$(W_{t+1}, H_{t+1}) = \begin{cases} (W_t, \mathcal{H}(W_t)) & \text{if } W_t \geq W^*(H_t) \\ (W_n(H_t), H_t) & \text{if } W_t < W^*(H_t) \end{cases}.$$

Moreover, there exists a housing level $H^s \in (H_0, H^o)$ such that $\mathcal{W}(H_t) < W^*(H_t) < W_n(H_t)$ for all $H_t \in [H_0, H^s)$ and $W^*(H_t) = W_n(H_t) = \mathcal{W}(H_t)$ for all $H_t \in [H^s, 1]$.

Proof of Fact A.1.7. We know that (W_{t+1}, H_{t+1}) solves problem (53). There are two possibilities to consider: i) the period t price constraint holds with equality at the optimal policies, and ii) the period t price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). If the period t price constraint holds with equality, then $(1-\beta)\mathcal{W}(H_{t+1}) = W_t - \beta W_{t+1}$. It then follows that $H_{t+1} = \mathcal{H}((W_t - \beta W_{t+1})/(1-\beta))$. The constraint that $H_{t+1} \geq H_t$ implies that $\mathcal{H}((W_t - \beta W_{t+1})/(1-\beta)) \geq H_t$ or equivalently that

$$\frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta} \geq W_{t+1}.$$

The constraint that $\mathcal{H}(W_{t+1}) \geq H_{t+1}$ implies that $W_{t+1} \geq W_t$. It follows that the range of feasible W_{t+1} values is

$$W_{t+1} \in [W_t, \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}].$$

For this interval to be non-empty it is necessary that $W_t \geq \mathcal{W}(H_t)$.

The optimal choice of period $t+1$ wealth must solve the problem

$$\max_{\{W_{t+1}\}} \left\{ \begin{array}{c} \mathcal{H}(\frac{W_t - \beta W_{t+1}}{1-\beta}) + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_{t+1}) \\ s.t.\ W_{t+1} \in [W_t, \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}] \end{array} \right\}.$$

The derivative of the objective function is

$$\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W_{t+1}) - \frac{\beta}{1-\beta}\mathcal{H}'(\frac{W_t - \beta W_{t+1}}{1-\beta}).$$

The concavity of the function $\mathcal{H}(W)$, implies that this derivative is negative for all $W_{t+1} \geq W_t$. The optimal choice of period $t+1$ wealth is therefore W_t. This in turn implies that $H_{t+1} = \mathcal{H}(W_t)$.

We conclude that if the period t price constraint holds with equality at the optimal policies, then the optimal policies are $(W_t, \mathcal{H}(W_t))$. A necessary condition for this to be the solution is that $W_t \geq \mathcal{W}(H_t)$. Note for future reference that the payoff from this candidate solution is

$$C + \frac{\mu\overline{\theta}}{1-\mu\beta}(\mathcal{H}(W_t) - 1) + \frac{\underline{u}}{1-\beta}. \tag{54}$$

Possibility ii). If the period t price constraint holds as an inequality at the optimal policies, then $(1-\beta)\mathcal{W}(H_{t+1}) > W_t - \beta W_{t+1}$ and $H_{t+1} = H_t$. This means that

$$W_{t+1} > \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}.$$

The constraint that $\mathcal{H}(W_{t+1}) \geq H_{t+1}$ requires that $W_{t+1} \geq \mathcal{W}(H_t)$. Define the wealth level $W_n(H_t)$ as the solution to the following problem

$$\max_{\{W_{t+1}\}} \left\{ \begin{array}{c} -\frac{\beta W_{t+1}}{H_t} + \mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_{t+1}) - 1)\right) \\ s.t.\ W_{t+1} \geq \mathcal{W}(H_t) \end{array} \right\}. \tag{55}$$

Then, the optimal policies must equal $(W_n(H_t), H_t)$ and it must be the case that

$$W_n(H_t) > \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}. \tag{56}$$

The payoff from this candidate solution is

$$C - \left(\frac{(1-\beta)\mathcal{W}(H_t) - (W_t - \beta W_n(H_t))}{H_t}\right) + \mu\overline{\theta}\left((H_t - 1) + \frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_n(H_t)) - 1)\right) + \frac{\underline{u}}{1-\beta}. \tag{57}$$

We now provide some more information about the wealth level $W_n(H_t)$. Let H^s be the housing level satisfying the following equality:

$$(1 - H^s)\,\overline{\theta} + S(H^s) + H^s S'(H^s) - C(1-\beta) = \underline{u} + H^s\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right).$$

Assumptions 1 and 2 imply that H^s is well-defined and lies between H_0 and H^o. Then we have the following claim:

Claim A.1.1. *If $H_t \geq H^s$, then $W_n(H_t) = \mathcal{W}(H_t)$ and if $H_t < H^s$, then $W_n(H_t) > \mathcal{W}(H_t)$.*

Proof of Claim A.1.1. By definition, the wealth level $W_n(H_t)$ solves problem (55). The derivative of the objective function in problem (55) is

$$-\frac{\beta}{H_t} + \mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W_{t+1}),$$

and the second derivative is

$$\mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}''(W_{t+1}) < 0.$$

The objective function is therefore concave. To prove the claim, it suffices to show that if $H_t \geq H^s$, then it is the case that

$$-\frac{\beta}{H_t} + \mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H_t)) \leq 0,$$

while if $H_t < H^s$, then it is the case that

$$-\frac{\beta}{H_t} + \mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H_t)) > 0.$$

Because the function $\mathcal{H}(W)$ is the inverse of the function $\mathcal{W}(H)$, we have that

$$\mathcal{H}'(W_{t+1}) = \frac{1}{\mathcal{W}'(\mathcal{H}(W_{t+1}))}.$$

Moreover, it is the case that

$$\mathcal{W}'(H) = \frac{C(1-\beta) + \underline{u} + H\overline{\theta} - (1-H)\overline{\theta} - S(H) - HS'(H)}{1-\beta}. \tag{58}$$

It follows that

$$\begin{aligned} &-\tfrac{\beta}{H_t} + \mu\overline{\theta}\tfrac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W_{t+1}) \\ = &-\tfrac{\beta}{H_t} + \mu\overline{\theta}\tfrac{\mu\beta}{1-\mu\beta}\left[\tfrac{1-\beta}{C(1-\beta)+\underline{u}+\mathcal{H}(W_{t+1})\overline{\theta}-(1-\mathcal{H}(W_{t+1}))\overline{\theta}-S(\mathcal{H}(W_{t+1}))-\mathcal{H}(W_{t+1})S'(\mathcal{H}(W_{t+1}))}\right]. \end{aligned}$$

In particular, therefore, we have that

$$\begin{aligned} &-\tfrac{\beta}{H_t} + \mu\overline{\theta}\tfrac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H_t)) \\ = &-\tfrac{\beta}{H_t} + \mu\overline{\theta}\tfrac{\mu\beta}{1-\mu\beta}\left[\tfrac{1-\beta}{C(1-\beta)+\underline{u}+H_t\overline{\theta}-(1-H_t)\overline{\theta}-S(H_t)-H_tS'(H_t)}\right]. \end{aligned} \tag{59}$$

The right hand side of (59) is non-positive if

$$\frac{\mu^2\overline{\theta}(1-\beta)}{1-\mu\beta}\left[\frac{1}{C(1-\beta)+\underline{u}+H_t\overline{\theta}-(1-H_t)\overline{\theta}-S(H_t)-H_tS'(H_t)}\right] \leq \frac{1}{H_t}.$$

This inequality is equivalent to

$$(1-H_t)\overline{\theta} + S(H_t) + H_tS'(H_t) - C(1-\beta) \leq \underline{u} + H_t\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right).$$

This follows from the fact that $H_t \geq H^s$. Similarly, the right hand side of (59) is positive if

$$(1-H_t)\overline{\theta} + S(H_t) + H_tS'(H_t) - C(1-\beta) > \underline{u} + H_t\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right),$$

and this follows from the fact that $H_t < H^s$. ■

Finally, note that it follows from the claim and the fact that the wealth level $W_n(H_t)$ solves problem (55) that when $H_t < H^s$ it must be the case that

$$\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(W_n(H_t)) = \frac{1}{H_t}. \tag{60}$$

Which possibility arises? Having understood the two possibilities, we can now analyze which one arises. Suppose first that $H_t \geq H^s$, so that $W_n(H_t) = \mathcal{W}(H_t)$. Then condition (56) implies that a necessary condition for possibility ii) to be the solution is that $W_t < \mathcal{W}(H_t)$. Furthermore, a necessary condition for possibility i) to be the solution is that $W_t \geq \mathcal{W}(H_t)$. Thus, we conclude that when $H_t \geq H^s$ the optimal policies are given by:

$$(W_{t+1}, H_{t+1}) = \begin{cases} (W_t, \mathcal{H}(W_t)) & \text{if } W_t \geq \mathcal{W}(H_t) \\ (\mathcal{W}(H_t), H_t) & \text{if } W_t < \mathcal{W}(H_t) \end{cases}. \tag{61}$$

The case in which $H_t < H^s$ is more complicated. It remains the case that a necessary condition for possibility i) to be the solution is that $W_t \geq \mathcal{W}(H_t)$ and condition (56) implies that a necessary condition for possibility ii) to be the solution is that $W_t < \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)$. Given that $W_n(H_t) > \mathcal{W}(H_t)$, we can conclude that the solution is $(W_n(H_t), H_t)$ if $W_t < \mathcal{W}(H_t)$ and $(W_t, \mathcal{H}(W_t))$ if $W_t \geq \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)$. For values of W_t in the interval $[\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$ both possibilities are feasible. Thus, which possibility is optimal depends on a comparison of the payoffs (54) and (57). We can show:

Claim A.1.2. *If $H_t < H^s$ there exists $\overline{W}(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$ such that the optimal policies are given by:*

$$(W_{t+1}, H_{t+1}) = \begin{cases} (W_t, \mathcal{H}(W_t)) & \textit{if } W_t \geq \overline{W}(H_t) \\ (W_n(H_t), H_t) & \textit{if } W_t < \overline{W}(H_t) \end{cases}. \tag{62}$$

Proof of Claim A.1.2. When $W_t \in [\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)]$, the solution will be $(W_t, \mathcal{H}(W_t))$ if (54) exceeds (57) and $(W_n(H_t), H_t)$ if (54) is less than (57). Differencing (54) and (57) yields

$$\begin{aligned} & \frac{\mu\overline{\theta}}{1-\mu\beta}\mathcal{H}(W_t) + \left(\frac{(1-\beta)\mathcal{W}(H_t) - (W_t - \beta W_n(H_t))}{H_t}\right) - \mu\overline{\theta}\left(H_t + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_n(H_t))\right) \\ = & \left(\frac{(1-\beta)\mathcal{W}(H_t) - (W_t - \beta W_n(H_t))}{H_t}\right) - \mu\overline{\theta}\left(H_t + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_n(H_t)) - \frac{1}{1-\mu\beta}\mathcal{H}(W_t)\right). \end{aligned}$$

Define the function $\varphi(W; H_t)$ on the interval $[\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)]$ to equal this difference.

Note first that

$$\varphi(\mathcal{W}(H_t); H_t) = \frac{\beta\left(W_n(H_t) - \mathcal{W}(H_t)\right)}{H_t} - \mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}\left(\mathcal{H}(W_n(H_t)) - \mathcal{H}(\mathcal{W}(H_t))\right)\right).$$

By the *Mean-Value Theorem,* there exists $W \in (\mathcal{W}(H_t), W_n(H_t))$ such that

$$\varphi(\mathcal{W}(H_t); H_t) = -\left[\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(W) - \frac{1}{H_t}\right]\beta\left(W_n(H_t) - \mathcal{W}(H_t)\right).$$

The concavity of the function $\mathcal{H}(W)$ then implies that

$$\varphi(\mathcal{W}(H_t); H_t) < -\left[\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(W_n(H_t)) - \frac{1}{H_t}\right]\beta\left(W_n(H_t) - \mathcal{W}(H_t)\right) = 0.$$

On the other hand, we have that

$$\begin{aligned}
& \varphi(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t); H_t) \\
= \; & -\mu\overline{\theta}\left(H_t + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_n(H_t)) - \frac{1}{1-\mu\beta}\mathcal{H}(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))\right) \\
= \; & \frac{\mu\overline{\theta}}{1-\mu\beta}\left[\mathcal{H}(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)) - \left(\left(1-\mu\beta\right)H_t + \mu\beta\mathcal{H}(W_n(H_t))\right)\right] \\
> \; & \frac{\mu\overline{\theta}}{1-\mu\beta}\left[\mathcal{H}(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)) - \left(\left(1-\beta\right)\mathcal{H}(\mathcal{W}(H_t)) + \beta\mathcal{H}(W_n(H_t))\right)\right] \\
> \; & 0,
\end{aligned}$$

where the first inequality follows from the fact that $H_t = \mathcal{H}(\mathcal{W}(H_t)) < \mathcal{H}(W_n(H_t))$ and the second inequality follows from the concavity of $\mathcal{H}(W)$.

Finally, we have that for all $W \in [\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)]$

$$\frac{\partial\varphi(W; H_t)}{\partial W} = -\frac{1}{H_t} + \mu\overline{\theta}\left(\frac{1}{1-\mu\beta}\mathcal{H}'(W)\right) > 0,$$

where the inequality follows from the concavity of $\mathcal{H}(W)$ and the fact that $W < W_n(H_t)$.

We conclude that there exists a unique $\overline{W}(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$ such that $\varphi(W_t; H_t) < 0$ if $W_t < \overline{W}(H_t)$ and $\varphi(W_t; H_t) > 0$ if $W_t > \overline{W}(H_t)$. ■

Pulling all this together, if we define the function

$$W^*(H_t) = \begin{cases} \overline{W}(H_t) & \textit{if } H_t < H^s \\ \mathcal{W}(H_t) & \textit{if } H_t \geq H^s \end{cases} \tag{63}$$

and remember that $W_n(H_t) = \mathcal{W}(H_t)$ if $H_t \geq H^s$, Fact A.1.7 follows from (61) and (62). ■

10.3 Verifying form of solution

We have now completed the characterization of the solution to the period t residents' problem and can verify it has the same form as described in Proposition 2. If $W_t \geq W^*(H_t)$, then, given that $H_{t+1} = \mathcal{H}(W_t)$, in period t the housing stock increases to $\mathcal{H}(W_t)$. Moreover, since

$g_{t+1} = (1-\delta)g^o(\mathcal{H}(W_t))$, the community invests in $g^o(H(W_t)) - g_t$ units of the public good in period t. Given that $W_{t+1} = W_t$, it must be the case that

$$c(1-\delta)g^o(\mathcal{H}(W_t)) - (1+\rho)b_{t+1} = cg_t - (1+\rho)b_t,$$

which implies that

$$(1+\rho)b_{t+1} - (1+\rho)b_t = c\left[(1-\delta)g^o(\mathcal{H}(W_t)) - g_t\right].$$

Thus, all but $c\delta g^o(\mathcal{H}(W_t))$ of the cost of investment is financed with debt. Since $H_\tau = \mathcal{H}(W_{t+1})$ and $g_\tau = (1-\delta)g^o(H_\tau)$ for all $\tau \geq t+2$, thereafter, the community maintains the public good at $g^o(\mathcal{H}(W_t))$ and the market provides no more housing. From (48) we have that $(1+\rho)b_\tau = (1+\rho)b_{t+1}$ for all $\tau \geq t+2$, implying that debt remains constant. This means that the community's wealth remains at W_t and taxes finance the maintenance of the public good and interest on the debt. The price of houses is C in period t and in all subsequent periods.

If $W_t < W^*(H_t)$, then, given that $H_{t+1} = H_t$, no new construction takes place in period t. Moreover, since $g_{t+1} = (1-\delta)g^o(H_t)$, the community invests in $g^o(H_t) - g_t$ units of the public good in period t. Given that $W_{t+1} = W_n(H_t)$, it must be the case that

$$c(1-\delta)g^o(H_t) - (1+\rho)b_{t+1} = W_n(H_t),$$

which implies that

$$(1+\rho)b_{t+1} - (1+\rho)b_t = c\left[(1-\delta)g^o(H_t) - g_t\right] - (W_n(H_t) - W_t).$$

The price of houses in period t is

$$C - \left(\frac{(1-\beta)\mathcal{W}(H_t) - (W_t - \beta W_n(H_t))}{H_t}\right),$$

which is less than C. Given that $H_{t+2} = \mathcal{H}(W_n(H_t))$, in period $t+1$ the housing stock increases to $\mathcal{H}(W_n(H_1))$. Moreover, since $g_{t+2} = (1-\delta)g^o(\mathcal{H}(W_n(H_t)))$, the community invests in $g^o(\mathcal{H}(W_n(H_t))) - (1-\delta)g^o(H_t)$ units of the public good in period t. From (48), we have that

$$(1+\rho)b_{t+2} - (1+\rho)b_{t+1} = c\left[(1-\delta)g^o(\mathcal{H}(W_n(H_t))) - (1-\delta)g^o(H_t)\right],$$

implying that all but $c\delta g^o(\mathcal{H}(W_n(H_t)))$ of the cost of investment is financed with debt. Since $H_\tau = \mathcal{H}(W_n(H_t))$and $g_\tau = (1-\delta)g^o(H_\tau)$ for all $\tau \geq t+3$, thereafter, the community maintains the public good at $g^o(\mathcal{H}(W_n(H_t)))$ and the market provides no more housing. From (48) we have that $(1+\rho)b_\tau = (1+\rho)b_{t+2}$ for all $\tau \geq t+3$, implying that debt remains constant. This means that the community's wealth remains at $W_n(H_t)$ and taxes finance the maintenance of the public good and interest on the debt. The price of houses is C in period $t+1$ and in all subsequent periods.

10.4 Properties of the functions $W^*(H_t)$ and $W_n(H_t)$

It only remains to establish that the functions $W^*(H_t)$ and $W_n(H_t)$ have the claimed properties. We begin with the function $W_n(H_t)$. From the proof of Fact A.1.7, we know that $W_n(H_t) = \mathcal{W}(H_t)$ if $H_t \geq H^s$ and that if $H_t < H^s$ it is the case that $W_n(H_t)$ satisfies (60). It follows immediately from the concavity of $\mathcal{H}(W)$ that $W_n(H_t)$ is increasing in H_t if $H_t < H^s$. Moreover, it is increasing if $H_t \geq H^s$, since $\mathcal{W}$ is increasing. The definition of H^s along with (60) imply that $\lim_{H_t \nearrow H^s} W_n(H_t) = \mathcal{W}(H^s)$. This implies that $W_n(H_t)$ is increasing on the entire interval $[H_0, 1]$ and is continuous.

We now turn to the function $W^*(H_t)$. This function is defined by (63). We know that if $H_t < H^s$, then $\overline{W}(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$. We also claim that $\overline{W}(H_t)$ is increasing for $H_t < H^s$. To see this, note that $\overline{W}(H_t)$ is implicitly defined by the equation $\varphi(\overline{W}(H_t); H_t) = 0$. It follows that

$$\frac{d\overline{W}}{dH_t} = -\frac{\frac{\partial \varphi(\overline{W}; H_t)}{\partial H_t}}{\frac{\partial \varphi(\overline{W}; H_t)}{\partial W}}.$$

We have already established that $\frac{\partial \varphi(\overline{W}; H_t)}{\partial W} > 0$. Thus, to establish the claim we need to show that $\frac{\partial \varphi(\overline{W}; H_t)}{\partial H_t} < 0$. Differentiating and using the first order condition (60), we have that

$$\begin{aligned}
\frac{\partial \varphi(\overline{W}; H_t)}{\partial H_t} &= \frac{(1-\beta)\mathcal{W}'(H_t)}{H_t} - \frac{\left[(1-\beta)\mathcal{W}(H_t) - \left(\overline{W} - \beta W_n(H_t)\right)\right]}{(H_t)^2} - \mu\overline{\theta} \\
&< \frac{(1-\beta)\mathcal{W}'(H_t)}{H_t} - \mu\overline{\theta},
\end{aligned}$$

where the inequality follows from the fact that

$$\frac{\left[(1-\beta)\mathcal{W}(H_t) - \left(\overline{W} - \beta W_n(H_t)\right)\right]}{(H_t)^2} > 0.$$

Using (58) and the definition of $\mathcal{W}(H_t)$, we have that

$$\frac{(1-\beta)\mathcal{W}'(H_t)}{H_t} = \frac{(1-\beta)\mathcal{W}(H_t)}{(H_t)^2} + \overline{\theta} - S'(H_t).$$

Thus,

$$\frac{\partial \varphi(\overline{W}; H_t)}{\partial H_t} < \frac{(1-\beta)\mathcal{W}(H_t)}{(H_t)^2} + \overline{\theta}(1-\mu) - S'(H_t).$$

It therefore suffices to show that

$$\frac{(1-\beta)\mathcal{W}(H_t)}{H_t} + H_t\overline{\theta}(1-\mu) - H_t S'(H_t) < 0.$$

Using the definition of $\mathcal{W}(H_t)$, we have that

$$\begin{aligned} & \frac{(1-\beta)\mathcal{W}(H_t)}{H_t} + H_t\overline{\theta}(1-\mu) - H_t S'(H_t) \\ = \quad & H_t\overline{\theta}(1-\mu) + C(1-\beta) + \underline{u} - (1-H_t)\overline{\theta} - S(H_t) - H_t S'(H_t). \end{aligned}$$

But since $H_t < H^s$, we have that

$$\begin{aligned} (1-H_t)\overline{\theta} - S(H_t) - H_t S'(H_t) \quad & > \quad C(1-\beta) + \underline{u} + H_t\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right) \\ & > \quad C(1-\beta) + \underline{u} + H_t\overline{\theta}\left(1-\mu\right), \end{aligned}$$

where the last inequality follows from the fact that

$$\frac{\mu^2(1-\beta)}{1-\mu\beta} < \mu.$$

Given that $\overline{W}(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$ and that $\lim_{H_t \nearrow H^s} W_n(H_t) = \mathcal{W}(H^s)$, it must also be the case that $\lim_{H_t \nearrow H^s} \overline{W}(H_t) = \mathcal{W}(H^s)$. Given that $W^*(H_t) = \mathcal{W}(H_t)$ if $H_t \geq H^s$, this implies that $W^*(H_t)$ is increasing on the entire interval $[H_0, 1]$ and is continuous.

11 Appendix 2: Proof of Theorem

Let $W^* \in \Psi$ and let $\xi(W^*)$ be the associated candidate equilibrium. We need to show that $\xi(W^*)$ is an equilibrium if W^* satisfies the conditions of the Theorem. This requires showing that the policy rules and value function defined by (25), (27), (28), (30), and (31) satisfy the conditions for equilibrium described in Section 5. Recall that there are two such conditions. The first is that, for all states (g, b, H), the policy rules solve the residents' problem (17) when the future price and continuation value are described by (30) and (31). The second is that, for all states (g, b, H), the value function satisfies the equality (18).

It is helpful to write out problem (17) with all its constraints as follows:

$$\max_{(g',b',T,H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right) - T + \beta V(g',b',H')\right] \\ s.t.\ (1+\rho)b + c\left(\frac{g'}{1-\delta} - g\right) = b' + H'T \\ g' \geq 0 \ \&\ H' \geq H \\ P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - T + \beta P(g',b',H') - \underline{u} \\ P \leq C\ (= \text{if } H' > H). \end{array} \right\}.$$

Solving the budget constraint for the tax T, substituting this into the objective function and market equilibrium condition, and using the notation W and W' to describe current and future wealth, we can remove the tax as a choice variable and write the problem as:

$$\max_{(g',W',H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta V(g',b',H')\right] \\ s.t.\ g' \geq 0 \ \&\ H' \geq H \\ P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} \\ + \frac{W - \beta W'}{H'} + \beta P(g',b',H') - \underline{u} \\ P \leq C\ (= \text{if } H' > H). \end{array} \right\} \tag{64}$$

While the wealth level W' replaces the debt level b' in the set of choice variables, the latter can be immediately recovered from the equation $b' = \beta\left(W' - cg'\right)$. This is the form of the residents' problem we will work with. Since behavior in our candidate equilibrium differs in states in which the housing stock is greater than H^s and states in which the housing stock is less than H^s, we consider the two situations separately. We begin with the former case, since it is simpler.

11.1 States (g, b, H) such that $H \geq H^s$

11.1.1 Policy rules and value function

The policy rules and value function depend on the functions $W^*(H)$, $H_c(W)$, and $W_n(H)$. Since $W^* \in \Psi$, we know that for states (g, b, H) such that $H \geq H^s$, $W^*(H) = \mathcal{W}(H)$. Moreover, in this range, the functions $H_c(W)$ and $W_n(H)$ turn out to be very simple. Our first result describes the function $H_c(W)$.

Fact A.2.1. *If $W \geq W^*(H)$ and $H \geq H^s$, then $H_c(W) = \mathcal{H}(W)$.*

Proof of Fact A.2.1. Using (26) and (29), in this range $H_c(W)$ is implicitly defined as the solution to the system

$$\mathcal{P}(H_c, \mathcal{W}(H_c), W) = C \ \& \ \mathcal{W}(H_c) \geq W. \tag{65}$$

From the definition of $\mathcal{W}(H)$, it is clear that

$$\mathcal{P}(\mathcal{H}(W), \mathcal{W}(\mathcal{H}(W)), W) = \mathcal{P}(\mathcal{H}(W), \mathcal{W}(\mathcal{H}(W)), \mathcal{W}(\mathcal{H}(W))) = C$$

and that $\mathcal{W}(\mathcal{H}(W)) = W$. This implies that $\mathcal{H}(W)$ is indeed a solution to (65). Suppose there is another solution, say, H'. Given the constraint that $\mathcal{W}(H') \geq W$, we know that H' exceeds $\mathcal{H}(W)$ and is thus greater than H^s. Given the definition of $\mathcal{W}(H)$, this implies that $\mathcal{P}(H', \mathcal{W}(H'), \mathcal{W}(H'))$ equals C. Thus, since $\mathcal{P}(H', \mathcal{W}(H'), W)$ must equal C, $W = \mathcal{W}(H')$ which contradicts the fact that H' exceeds $\mathcal{H}(W)$. ■

Given Fact A.2.1, (25) implies that the equilibrium public good and housing rules are:

$$(g'(g, b, H), H'(g, b, H)) = \begin{cases} ((1-\delta)g^o(H), H) \text{ if } W < \mathcal{W}(H) \\ ((1-\delta)g^o(\mathcal{H}(W)), \mathcal{H}(W)) \text{ if } W \geq \mathcal{W}(H) \end{cases}. \tag{66}$$

Moreover, from (31) and (32), the value function is

$$V(g, b, H) = \begin{cases} V^*(\mathcal{W}(H)) + \frac{W - \mathcal{W}(H)}{H} & \text{if } W < \mathcal{W}(H) \\ V^*(W) & \text{if } W \geq \mathcal{W}(H) \end{cases}, \tag{67}$$

where

$$V^*(W) = C + \frac{u}{1-\beta} + \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\right)(\mathcal{H}(W) - 1). \tag{68}$$

It is clear that V^* is increasing, since

$$\frac{dV^*(W)}{dW} = \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\right)\mathcal{H}'(W) > 0. \tag{69}$$

Moreover, V^* is strictly concave, since

$$\frac{d^2V^*(W)}{dW^2} = \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\right)\mathcal{H}''(W) < 0. \tag{70}$$

Importantly, it is also the case that

$$\mu\frac{dV^*(\mathcal{W}(H^s))}{dW} = \frac{1}{H^s}. \tag{71}$$

and that for all $H > H^s$

$$\mu\frac{dV^*(\mathcal{W}(H))}{dW} < \frac{1}{H}. \tag{72}$$

To understand these findings note that

$$\mu\frac{dV^*(\mathcal{W}(H))}{dW} - \frac{1}{H} = \left(\frac{\mu}{1-\mu\beta}\right)\mu\overline{\theta}\mathcal{H}'(\mathcal{W}(H)) - \frac{1}{H}.$$

As shown in the proof of Fact A.1.7 (see proof of Claim A.1.1), this derivative equals

$$\left(\frac{\mu}{1-\mu\beta}\right)\mu\overline{\theta}\left[\frac{1-\beta}{C(1-\beta)+\underline{u}+H\overline{\theta}-(1-H)\overline{\theta}-S\left(H\right)-HS'\left(H\right)}\right] - \frac{1}{H}.$$

We know from (24) that, by definition,

$$\underline{u}+H^s\overline{\theta}-H^s\overline{\theta}\left(\frac{\mu^2(1-\beta)}{1-\mu\beta}\right) = (1-H^s)\overline{\theta}+S\left(H^s\right)+H^sS'\left(H^s\right)-C(1-\beta).$$

Rearranging this, yields

$$\left(\frac{\mu}{1-\mu\beta}\right)\mu\overline{\theta}\left[\frac{1-\beta}{C(1-\beta)+\underline{u}+H^s\overline{\theta}-(1-H^s)\overline{\theta}-S\left(H^s\right)-H^sS'\left(H^s\right)}\right] = \frac{1}{H^s},$$

which establishes (71). For (72), it is enough to show that for all $H > H^s$,

$$\underline{u}+H\overline{\theta}-H\overline{\theta}\left(\frac{\mu^2(1-\beta)}{1-\mu\beta}\right) > (1-H)\overline{\theta}+S\left(H\right)+HS'\left(H\right)-C(1-\beta). \tag{73}$$

This follows from (24) and Assumption 1(i).

Our next result describes the function $W_n(H)$.

Fact A.2.2. *If $H \geq H^s$, then $W_n(H) = \mathcal{W}(H)$.*

Proof of Fact A.2.2. Using (29) and (33), we have that

$$W_n(H) \equiv \arg\max_{W'}\left\{\begin{array}{c}\frac{\underline{u}}{1-\beta}+\mathcal{P}(H,W',\mathcal{W}(H))+\mu\overline{\theta}(H-1)+\mu V^*(W') \\ s.t.\ \mathcal{P}(H,W',\mathcal{W}(H)) \leq C\end{array}\right\}. \tag{74}$$

The constraint in this problem is slack for all $W' \geq \mathcal{W}(H)$ and is violated for all $W' < \mathcal{W}(H)$. Thus, if the objective function is decreasing in W' for all $W' \geq \mathcal{W}(H)$, it must be the case that $W_n(H) = \mathcal{W}(H)$. The derivative of the objective function is

$$\mu \frac{dV^*(W')}{dW} - \frac{1}{H}.$$

From (70) and (72), we have that

$$\mu \frac{dV^*(W')}{dW} - \frac{1}{H} \leq \mu \frac{dV^*(\mathcal{W}(H))}{dW} - \frac{1}{H} < 0,$$

as required. ■

Given Facts A.2.1 and A.2.2, (27) implies that the equilibrium debt rule is

$$b'(g, b, H) = \begin{cases} \frac{c(1-\delta)g^o(H) - \mathcal{W}(H)}{1+\rho} & \text{if } W < \mathcal{W}(H) \\ \frac{c(1-\delta)g^o(\mathcal{H}(W)) - W}{1+\rho} & \text{if } W \geq \mathcal{W}(H) \end{cases}. \tag{75}$$

This, together with the public good rule, imply that next period's wealth W' is $\mathcal{W}(H)$ when $W < \mathcal{W}(H)$ and W when $W \geq \mathcal{W}(H)$. The price rule is

$$P(g, b, H) = \begin{cases} \mathcal{P}(H, \mathcal{W}(H), W) & \text{if } W < \mathcal{W}(H) \\ C & \text{if } W \geq \mathcal{W}(H) \end{cases}. \tag{76}$$

11.1.2 The residents' problem

We are now ready to show that the policy rules defined by (66), (75), (76) solve the residents' problem (64) given that the future price and continuation value are as described in (76) and (67) and (68). Our first observation is that we can assume that the residents' policy choices are such that next period's wealth is at least as big as the threshold $\mathcal{W}(H')$.

Fact A.2.3. *Suppose that $H \geq H^s$ and let (g', W', H', P) solve problem (64) with future price and continuation value as described by (76) and (67) and (68). Then, we may assume without loss of generality that $W' \geq \mathcal{W}(H')$.*

Proof of Fact A.2.3. Suppose that $W' < \mathcal{W}(H')$. We will show that increasing W' to $\mathcal{W}(H')$ will not violate the constraints or change the value of the objective function in problem (64). Regarding the former, note that using the equilibrium price rule, the definition of $\mathcal{P}$ in (29), and the definition of $\mathcal{W}(H')$ in (20), we have

$$\begin{aligned}
P &= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta P(g',b',H') - \underline{u} \\
&= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta \mathcal{P}(H',\mathcal{W}(H'),W') - \underline{u} \\
&= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta \mathcal{W}(H')}{H'} + \beta \mathcal{P}(H',\mathcal{W}(H'),\mathcal{W}(H')) - \underline{u} \\
&= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta \mathcal{W}(H')}{H'} + \beta C - \underline{u}.
\end{aligned}$$

Regarding the latter, note that using the equilibrium value function and the definition of $\mathcal{W}(H')$ in (20), we have

$$\begin{aligned}
& B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta V(g',b',H') \\
=\ & B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta\left[V^*(\mathcal{W}(H')) + \frac{W'-\mathcal{W}(H')}{H'}\right] \\
=\ & B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta \mathcal{W}(H')}{H'} + \beta V^*(\mathcal{W}(H')).
\end{aligned}$$

■

Fact A.2.3 simplifies matters considerably as it ties down the form of the value function and fixes the future housing price at C. It allows us to rewrite problem (64) as

$$\max_{(g',W',H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P+\frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta V^*(W')\right] \\ s.t.\ g' \geq 0 \ \&\ H' \geq H \\ P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta C - \underline{u} \\ P \leq C\ (=\text{if } H' > H) \\ W' \geq \mathcal{W}(H') \end{array} \right\}. \tag{77}$$

Our next result shows that public good provision is efficient.

Fact A.2.4. *Suppose that $H \geq H^s$ and let (g',W',H',P) solve problem (77) with $V^*(W')$ given by (68). Then, $g' = (1-\delta)g^o(H')$.*

Proof of Fact A.2.4. Note first that

$$(1-\delta)g^o(H') = \arg\max_{g'} \left\{ B\left(\frac{g'/(1-\delta)}{(H')^\alpha}\right) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} \right\}.$$

Thus, provided that such a choice does not make P greater than C, it will clearly be optimal to set g' equal to $(1-\delta)g^o(H')$. Suppose then that such a choice does violate the price constraint; i.e.,

$$(1-H')\overline{\theta} + S(H') + \frac{W - \beta W'}{H'} + \beta C - \underline{u} > C.$$

Then clearly it must be the case that the price constraint binds under the policies (g', W', H'), which means that

$$C = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta C - \underline{u}.$$

This implies that the payoff from the policies (g', W', H') is

$$(1-\mu\beta)\left(C + \frac{\underline{u}}{1-\beta}\right) + \mu\overline{\theta}(H'-1) + \mu\beta V^*(W').$$

Now choose $\widehat{W}$ greater than W' to satisfy the price constraint with the efficient level of the public good; i.e.,

$$(1-H')\overline{\theta} + S(H') + \frac{W - \beta \widehat{W}}{H'} + \beta C - \underline{u} = C.$$

Consider the alternative policies $((1-\delta)g^o(H'), \widehat{W}, H')$ which involve the efficient level of public good and a higher level of wealth passed to next period's residents. Clearly, these policies satisfy the constraints. Moreover, the payoff from them is

$$(1-\mu\beta)\left(C + \frac{\underline{u}}{1-\beta}\right) + \mu\overline{\theta}(H'-1) + \mu\beta V^*(\widehat{W}).$$

This exceeds the payoff from the policies (g', W', H') since $V^*(\cdot)$ is increasing in wealth (see (69)). ■

Fact A.2.4 allows us to write the residents' problem as

$$\max_{(W',H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H') + \frac{W-\beta W'}{H'} + \beta V^*(W')\right] \\ s.t.\ H' \geq H \\ P = (1-H')\overline{\theta} + S(H') + \frac{W-\beta W'}{H'} + \beta C - \underline{u} \\ P \leq C \ (= \text{if } H' > H) \\ W' \geq \mathcal{W}(H') \end{array} \right\}$$

or, eliminating P, as

$$\max_{(W',H')}\left\{\begin{array}{c}(1-\mu)\left[(1-H')\overline{\theta}+\beta C-\underline{u}+\frac{\underline{u}}{1-\beta}\right]+S(H')+\frac{W-\beta W'}{H'}+\mu\beta V^{*}(W')\\ s.t.\ H'\geq H\\ (1-H')\overline{\theta}+S(H')+\frac{W-\beta W'}{H'}-(1-\beta)C\leq\underline{u}\ (=\text{if } H'>H)\\ W'\geq\mathcal{W}(H')\end{array}\right\} \tag{78}$$

Our next result ties down the optimal (W', H').

Fact A.2.5. *Suppose that $H \geq H^s$ and let (W', H') solve problem (78) with $V^*(W')$ given by (68). Then,*

$$(W',H')=\left\{\begin{array}{ll}(\mathcal{W}(H),H) & \text{if } W<\mathcal{W}(H)\\ (W,\mathcal{H}(W)) & \text{if } W\geq\mathcal{W}(H)\end{array}\right. .$$

Proof of Fact A.2.5. There are two possibilities to consider: i) the price constraint holds with equality at the optimal policies, and ii) the price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). If the price constraint holds with equality, then $(1-\beta)\mathcal{W}(H') = W - \beta W'$. It then follows that $H' = \mathcal{H}((W-\beta W')/(1-\beta))$. The constraint that $H' \geq H$ implies that $\mathcal{H}((W-\beta W')/(1-\beta)) \geq H$ or equivalently that

$$\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}\geq W'.$$

The constraint that $W' \geq \mathcal{W}(H')$ implies that $\mathcal{H}(W') \geq H' = \mathcal{H}((W-\beta W')/(1-\beta))$ and hence that $W' \geq W$. It follows that the range of feasible W' values is

$$W'\in[W,\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}].$$

For this interval to be non-empty it is necessary that $W \geq \mathcal{W}(H)$.

Using (68), the optimal choice of wealth must solve the problem

$$\max_{\{W'\}}\left\{\begin{array}{c}\mathcal{H}(\frac{W-\beta W'}{1-\beta})+\frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W')\\ s.t.\ W'\in[W,\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]\end{array}\right\}.$$

The derivative of the objective function in this problem is

$$\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W')-\frac{\beta}{1-\beta}\mathcal{H}'(\frac{W-\beta W'}{1-\beta}).$$

The concavity of the function $\mathcal{H}(W)$, implies that this derivative is negative for all $W' \geq W$. The optimal choice of wealth is therefore W. This in turn implies that $H' = \mathcal{H}(W)$.

We conclude that if the price constraint holds with equality at the optimal policies, then the optimal policies are $(W, \mathcal{H}(W))$. A necessary condition for this to be the solution is that $W \geq \mathcal{W}(H)$.

Possibility ii). If the price constraint holds as an inequality at the optimal policies, then $(1-\beta)\mathcal{W}(H') > W - \beta W'$ and $H' = H$. This means that

$$W' > \frac{W - (1-\beta)\mathcal{W}(H)}{\beta}.$$

The constraint that $W' \geq \mathcal{W}(H')$ requires that $W' \geq \mathcal{W}(H)$. Using (68), the optimal choice of wealth must solve the problem

$$\max_{\{W'\}} \left\{ \begin{array}{c} -\frac{\beta W'}{H} + \mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W') - 1)\right) \\ s.t.\ W' \geq \mathcal{W}(H) \end{array} \right\}.$$

This problem is identical to problem (74) and hence, by Fact A.2.2, the solution is $\mathcal{W}(H)$. For the price constraint to hold as an inequality it must be the case that

$$\mathcal{W}(H) > \frac{W - (1-\beta)\mathcal{W}(H)}{\beta},$$

which requires that $W < \mathcal{W}(H)$.

We conclude that if the price constraint holds as an inequality at the optimal policies, then the optimal policies are $(\mathcal{W}(H), H)$. A necessary condition for this to be the solution is that $W < \mathcal{W}(H)$.

Which possibility arises? Having understood the two possibilities, we can now analyze which one arises. A necessary condition for possibility ii) to be the solution is that $W < \mathcal{W}(H)$. Furthermore, a necessary condition for possibility i) to be the solution is that $W \geq \mathcal{W}(H)$. Thus, we conclude that the optimal policies are given by:

$$(W', H') = \begin{cases} (\mathcal{W}(H), H) & \text{if } W < \mathcal{W}(H) \\ (W, \mathcal{H}(W)) & \text{if } W \geq \mathcal{W}(H) \end{cases},$$

as required. $\blacksquare$

Using Facts A.2.4 and A.2.5, it is clear that the residents want to follow the equilibrium policy rules described in (66), (75), and (76).

11.1.3 Verifying form of value function

It remains to verify that the value function as described by (67) and (68) satisfies (18). Suppose first that (g, b, H) is such that $W \geq \mathcal{W}(H)$. Then, we have that

$$
\begin{aligned}
& (1-\mu)\left[P(\cdot) + \frac{u}{1-\beta}\right] + \mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right) - T(\cdot) + \beta V(g'(\cdot), b'(\cdot), H'(\cdot))\right] \\
= \quad & (1-\mu)\left[C + \frac{u}{1-\beta}\right] + \mu\left[S(\mathcal{H}(W)) + \frac{W - \beta W}{\mathcal{H}(W)} + \beta V^*(W)\right] \\
= \quad & V^*(W),
\end{aligned}
$$

as required. Now suppose that (g, b, H) is such that $W < \mathcal{W}(H)$. Then, we have that

$$
\begin{aligned}
& (1-\mu)\left[P(\cdot) + \frac{u}{1-\beta}\right] + \mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right) - T(\cdot) + \beta V(g'(\cdot), b'(\cdot), H'(\cdot))\right] \\
= \quad & (1-\mu)\left[\mathcal{P}(H, \mathcal{W}(H), W) + \frac{u}{1-\beta}\right] + \mu\left[S(H) + \frac{W - \beta\mathcal{W}(H)}{H} + \beta V^*(\mathcal{W}(H))\right] \\
= \quad & (1-\mu)\left[C + \frac{u}{1-\beta}\right] + \mu\left[S(H) + \frac{\mathcal{W}(H) - \beta\mathcal{W}(H)}{H} + \beta V^*(\mathcal{W}(H))\right] + \frac{W - \mathcal{W}(H)}{H} \\
= \quad & V^*(\mathcal{W}(H)) + \frac{W - \mathcal{W}(H)}{H},
\end{aligned}
$$

as required.

11.2 States (g, b, H) such that $H < H^s$

11.2.1 Policy rules and value function

The functions $H_c(W)$ and $W_n(H)$ are defined by (26) and (33) and are more complicated in this range. We present two key results that establish some important properties of these functions. The first result concerns the function $H_c(W)$.

Fact A.2.6 *If $W \in [W^*(H), \mathcal{W}(H^s))$ and $H < H^s$, then $H_c(W)$ is uniquely defined, belongs to the interval (H, H^s), and is increasing in W. Moreover, for any initial housing level $H_0 < H^s$, the sequence $\langle H_t \rangle_{t=1}^{\infty}$ defined inductively by $H_t = H_c(W^*(H_{t-1}))$ converges monotonically to H^s. Similarly, for any initial wealth level $W_0 < \mathcal{W}(H^s)$, the sequence $\langle W_t \rangle_{t=1}^{\infty}$ defined inductively by $W_t = W^*(H_c(W_{t-1})))$ converges monotonically to $\mathcal{W}(H^s)$.*

Proof of Fact A.2.6 Using (26) and (29), $H_c(W)$ satisfies the equation $\mathcal{P}(H_c, W^*(H_c), W) = C$. Because $W \in [W^*(H), \mathcal{W}(H^s))$ and $W^* \in \Psi$ we know from the assumed properties of $W^*(H)$ that

$$\mathcal{P}(H, W^*(H), W) \geq \mathcal{P}(H, W^*(H), W^*(H)) > C,$$

and that

$$\mathcal{P}(H^s, W^*(H^s), W) < \mathcal{P}(H^s, W^*(H^s), W^*(H^s)) = C.$$

Thus, by the *Intermediate Value Theorem*, there must exist a solution $H_c \in (H, H^s)$ at which $\mathcal{P}(H_c, W^*(H_c), W) = C$. Moreover, at this solution, we must have that $W^*(H_c) > W$. If not, then $W^*(H_c) \leq W$. But then we have that

$$C = \mathcal{P}(H_c, W^*(H_c), W) \geq \mathcal{P}(H_c, W^*(H_c), W^*(H_c)) > C,$$

which is a contradiction.

For uniqueness, it is sufficient that

$$\frac{d\mathcal{P}(H, W^*(H), W)}{dH} < 0$$

at any solution of the equation $\mathcal{P}(H, W^*(H), W) = C$. Note from (29), that

$$\frac{d\mathcal{P}(H, W^*(H), W)}{dH} = -\overline{\theta} + S'(H) - \frac{\beta}{H}\frac{dW^*(H)}{dH} - \frac{(W - \beta W^*(H))}{H^2}.$$

Moreover, at a solution

$$-\frac{(W - \beta W^*(H))}{H} = \left[(1-H)\overline{\theta} + S(H) - C(1-\beta) - \underline{u}\right].$$

Thus, at a solution

$$\begin{aligned}\frac{d\mathcal{P}(H, W^*(H), W)}{dH} &= -\overline{\theta} + S'(H) - \frac{\beta}{H}\frac{dW^*(H)}{dH} + \frac{\left[(1-H)\overline{\theta} + S(H) - C(1-\beta) - \underline{u}\right]}{H} \\ &= -\left[\frac{C(1-\beta) + \underline{u} - (1-2H)\overline{\theta} - HS'(H) - S(H) + \beta\frac{dW^*(H)}{dH}}{H}\right].\end{aligned}$$

Given that $W^*(H)$ is increasing, we have

$$\frac{d\mathcal{P}(H, W^*(H), W)}{dH} < -\left[\frac{\underline{u} + H\overline{\theta} - \left((1-H)\overline{\theta} + S(H) + HS'(H) - C(1-\beta)\right)}{H}\right] < 0$$

where the last inequality follows from Assumptions 1(i) and 2.

A similar logic implies that $H_c(W)$ is increasing in W. Given that the solution satisfies $C = \mathcal{P}(H_c, W^*(H_c), W)$, we have that

$$\frac{dH_c}{dW} = -\frac{\frac{\partial\mathcal{P}(H_c, W^*(H_c), W)}{\partial W}}{\frac{d\mathcal{P}(H_c, W^*(H_c), W)}{dH}} = -\frac{1}{H_c\frac{d\mathcal{P}(H_c, W^*(H_c), W)}{dH}}.$$

Given that $d\mathcal{P}(H_c, W^*(H_c), W)/dH$ is negative, the result follows.

Now let $H_0 < H^s$ and consider the sequence $\langle H_t \rangle_{t=0}^{\infty}$ defined inductively by $H_t = H_c(W^*(H_{t-1}))$. We first show the sequence is increasing. To see this, recall that we showed that for any $H < H^s$, if $W \in [W^*(H), \mathcal{W}(H^s))$, then $H_c(W)$ belongs to the interval (H, H^s). Taking $H = H_{t-1}$

and $W = W^*(H_{t-1})$, this implies that $H_c(W^*(H_{t-1}))$ belongs to the interval (H_{t-1}, H^s). It remains to show that the sequence converges to H^s. Since the sequence is bounded by H^s and is increasing, it converges. Let H_∞ denote this limit. We know that for all $t \geq 1$, we have that $\mathcal{P}(H_t, W^*(H_t), W^*(H_{t-1})) = C$. Given that $W^*(H)$ is continuous, it follows that $\mathcal{P}(H_\infty, W^*(H_\infty), W^*(H_\infty)) = C$. Since $\mathcal{P}(H, W^*(H), W^*(H)) > C$ for all $H < H^s$, this implies that H_∞ must equal H^s (recall that $\mathcal{P}(H^s, W^*(H^s), W^*(H^s)) = C$).

Finally, let $W_0 < \mathcal{W}(H^s)$ and consider the sequence $\langle W_t \rangle_{t=1}^{\infty}$ defined inductively by $W_t = W^*(H_c(W_{t-1})))$. Associated with this sequence of wealth levels, is a sequence of housing levels $\langle H_t \rangle_{t=0}^{\infty}$ defined inductively by $H_t = H_c(W_{t-1})$. This sequence satisfies the equation $H_t = H_c(W^*(H_{t-1}))$ and hence converges monotonically to H^s. Given that the function $W^*(H)$ is increasing and $W^*(H^s)$ equals $\mathcal{W}(H^s)$, it follows that the sequence $\langle W_t \rangle_{t=1}^{\infty}$ converges monotonically to $W(H^s)$. ■

We can use Fact A.2.6 to shed light on the form of the function $V^*(W)$ defined in (32). For any $W < \mathcal{W}(H^s)$, we can construct a sequence $\langle W_t(W) \rangle_{t=0}^{\infty}$ that starts at W (i.e., $W_0(W) = W$) and is defined inductively by $W_t(W) = W^*(H_c(W_{t-1}(W))))$ as in Fact A.2.6. We can then write

$$V^*(W) = (1-\mu)\left[C + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S\left(H_c(W_0(W))\right) + \frac{W_0(W) - \beta W_1(W)}{H_c(W_0(W))} + \beta V^*(W_1(W))\right].$$

Similarly, we can write

$$V^*(W_1(W)) = (1-\mu)\left[C + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S\left(H_c(W_1(W))\right) + \frac{W_1(W) - \beta W_2(W)}{H_c(W_1(W))} + \beta V^*(W_2(W))\right].$$

Iterating, we obtain

$$V^*(W) = \sum_{t=0}^{\infty} (\mu\beta)^t \left((1-\mu)\left[C + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S\left(H_c(W_t(W))\right) + \frac{W_t(W) - \beta W_{t+1}(W)}{H_c(W_t(W))}\right]\right).$$

We know from the definition of the function $H_c(W)$ that for all t

$$(1 - H_c(W_t(W)))\overline{\theta} + S\left(H_c(W_t(W))\right) + \frac{W_t(W) - \beta W_{t+1}(W)}{H_c(W_t(W))} - C(1-\beta) = \underline{u}.$$

This implies that $H_c(W_t(W)) = \mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})$ and allows us to write

$$V^*(W) = C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1\right]. \tag{79}$$

Combining (68) and (79), we may conclude that

$$V^*(W) = \begin{cases} C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1\right] & \text{if } W \in [W^*(H_0), \mathcal{W}(H^s)) \\ C + \frac{\underline{u}}{1-\beta} + \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\right)(\mathcal{H}(W) - 1) & \text{if } W \in [\mathcal{W}(H^s), \mathcal{W}(1)] \end{cases} \tag{80}$$

There are several points to note about the $V^*(W)$ function. First, note that it is continuous at $W = \mathcal{W}(H^s)$, since for all t

$$\lim_{W \nearrow \mathcal{W}(H^s)} W_t(W) - \beta W_{t+1}(W) = (1-\beta)\mathcal{W}(H^s).$$

Second, the function is increasing. We have already demonstrated that this is true on the interval $[\mathcal{W}(H^s), \mathcal{W}(1)]$ (see (69)). To see that it is true on the interval $[W^*(H_0), \mathcal{W}(H^s))$, it is easiest to write the function as

$$V^*(W) = C + \frac{u}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \left[H_c(W_t(W)) - 1\right].$$

It follows that

$$\frac{dV^*(W)}{dW} = \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \frac{dH_c(W_t(W))}{dW} W_t'(W).$$

As shown in the proof of Fact A.2.6, $dH_c(W_t(W))/dW$ is positive. We claim that for all t, $W_t'(W)$ is also positive. The proof is by induction. We know that $W_0(W) = W$ and hence the result is true for $t = 0$. Now suppose the result is true for all $\tau \leq t-1$ and consider if it is true for t. By definition, we have that $W_t(W) = W^*(H_c(W_{t-1}(W)))$. Thus, we have that

$$W_t'(W) = \frac{dW^*(H_c(W_{t-1}(W))))}{dH} \frac{dH_c(W_{t-1}(W))}{dW} W_{t-1}'(W).$$

Each term on the right hand side of this expression is positive, and hence $W_t'(W)$ is positive as required.

Third, the value function is likely to be kinked at $W = \mathcal{W}(H^s)$. This is because

$$\frac{dV^*(W)}{dW} = \begin{cases} \mu\overline{\theta} \sum_{t=0}^{\infty} (\mu\beta)^t \mathcal{H}'(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})(\frac{W_t'(W) - \beta W_{t+1}'(W)}{1-\beta}) \text{ if } W \in [W^*(H_0), \mathcal{W}(H^s)) \\ \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\right) \mathcal{H}'(W) \text{ if } W \in [\mathcal{W}(H^s), \mathcal{W}(1)] \end{cases}$$

Under the assumption that $V^*(W)$ is concave, it follows from (71) that for all $W \in [W^*(H_0), \mathcal{W}(H^s))$[36]

$$\mu\frac{dV^*(W)}{dW} > \frac{1}{H^s}.$$

Finally, note that for all $W \in [W^*(H_0), \mathcal{W}(H^s))$, the second derivative of the function $V^*(W)$ is

$$\frac{d^2V^*(W)}{dW^2} = \mu\overline{\theta} \sum_{t=0}^{\infty} (\mu\beta)^t \begin{bmatrix} \mathcal{H}''(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})(\frac{W_t'(W) - \beta W_{t+1}'(W)}{1-\beta})^2 \\ +\mathcal{H}'(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})(\frac{W_t''(W) - \beta W_{t+1}''(W)}{1-\beta}) \end{bmatrix}.$$

[36] It can be shown analytically (see the proof of Fact A.4.1) that, if the value function is concave and kinked at $\mathcal{W}(H^s)$, then the left hand derivative of the value function at $\mathcal{W}(H^s)$ must be equal to $\mu(1-\mu\beta)/[(1-\beta)H^s]$. Accordingly, a necessary condition for concavity is that β exceed $1/(1+\mu)$. For realistic parameterizations, this condition will be always be satisfied.

The first part of the terms that is summed is clearly negative, but the second part is harder to sign. This is why it is difficult to establish analytically that the $V^*(W)$ function in this range is concave. Our assumption that $V^*(W)$ is concave amounts to assuming that the second part, if positive, does not fully offset the first part.

The second fact concerns the function $W_n(H)$.

Fact A.2.7. *If $H < H^s$, then $W_n(H) > W^*(H)$. Moreover, $\mathcal{P}(H, W_n(H), W^*(H)) < C$.*

Proof of Fact A.2.7. We begin with the first claim. By definition (33), we know that $\mathcal{P}(H, W_n(H), W^*(H))$ must be less than or equal to C. Suppose, to the contrary, that $W_n(H) \leq W^*(H)$. Then, since $W^* \in \Psi$, we have

$$\mathcal{P}(H, W_n(H), W^*(H)) \geq \mathcal{P}(H, W^*(H), W^*(H)) > C.$$

This is a contradiction.

Turning to the second claim, we will show that if $\mathcal{P}(H, W_n(H), W^*(H)) = C$ it must be that the payoff from the policies $(W_n(H), H)$ when the state is $(W^*(H), H)$ is strictly lower than the payoff from the equilibrium policies $(W^*(H_c(W^*(H))), H_c(W^*(H)))$. But this is inconsistent with equality (34) being satisfied at H, which is a contradiction. This will imply that it must be the case that $\mathcal{P}(H, W_n(H), W^*(H)) < C$.

Consider first the payoff from the policies $(W_n(H), H)$ when the state is $(W^*(H), H)$. By the first claim, we know that $W_n(H) > W^*(H)$ so that there will be new construction next period. It then follows that next period's price will be $\mathcal{P}(H, W^*(H_c(W_n(H))), W_n(H)) = C$. Moreover, the payoff from $(W_n(H), H)$ can be written as

$$(1-\mu)\left[(1-H)\overline{\theta} + \beta C + \underline{u}\frac{\beta}{1-\beta}\right] + S\left(H\right) + \frac{W^*(H) - \beta W_n(H)}{H} + \mu\beta V^*(W_n(H)). \tag{81}$$

Since $\mathcal{P}(H, W_n(H), W^*(H))$ is equal to C, we have that

$$S\left(H\right) + \frac{W^*(H) - \beta W_n(H)}{H} = C(1-\beta) + \underline{u} - (1-H)\overline{\theta}.$$

We can therefore write (81) as

$$C(1-\mu\beta) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} + \mu\overline{\theta}(H-1) + \mu\beta V^*(W_n(H)).$$

Now note that $H = \mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta})$, so we can write this as

$$C(1-\mu\beta) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta}) - 1\right] + \mu\beta V^*(W_n(H)). \tag{82}$$

Next observe from (79) that

$$V^*(W_n(H)) = C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \left[\mathcal{H}(\frac{W_t(W_n(H)) - \beta W_{t+1}(W_n(H))}{1-\beta}) - 1 \right].$$

Substituting this into (82), the payoff from $(W_n(H), H)$ can be written as

$$C + \frac{\underline{u}}{1-\beta} + \mu\overline{\theta} \left[\mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta}) - 1 \right] + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mu\overline{\theta} \left[\mathcal{H}(\frac{W_t(W_n(H)) - \beta W_{t+1}(W_n(H))}{1-\beta}) - 1 \right]$$

Note also that the payoff from the equilibrium policies $(W^*(H_c(W^*(H))), H_c(W^*(H)))$ in state $(W^*(H), H)$ can be written in a similar way as

$$\begin{aligned} & C + \frac{\underline{u}}{1-\beta} + \mu\overline{\theta} \left[\mathcal{H}(\frac{W^*(H) - \beta W^*(H_c(W^*(H)))}{1-\beta}) - 1 \right] \\ & + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mu\overline{\theta} \left[\mathcal{H}(\frac{W_t(W^*(H_c(W^*(H)))) - \beta W_{t+1}(W^*(H_c(W^*(H))))}{1-\beta}) - 1 \right] \end{aligned}$$

Moreover, we know that since

$$H = \mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta}) < H_c(W^*(H)) = \mathcal{H}(\frac{W^*(H) - \beta W^*(H_c(W^*(H)))}{1-\beta}),$$

we must have that $W_n(H) > W^*(H_c(W^*(H)))$.

Now define the function $\varphi(W)$ on the interval $[W^*(H_c(W^*(H))), W_n(H)]$ as follows:

$$\varphi(W) = C + \frac{\underline{u}}{1-\beta} + \mu\overline{\theta} \left[\mathcal{H}(\frac{W^*(H) - \beta W}{1-\beta}) - 1 \right] + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mu\overline{\theta} \left[\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1 \right]$$

If we can show that this function is decreasing in W, we will have established that $(W_n(H), H)$ must yield a smaller payoff than $(W^*(H_c(W^*(H))), H_c(W^*(H)))$ in state $(W^*(H), H)$, which contradicts equality (34).

Consider then differentiating $\varphi(W)$. Ignoring multiplicative constants, the derivative is

$$-\mathcal{H}'(\frac{W^*(H) - \beta W}{1-\beta})\beta + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mathcal{H}'(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})(W_t'(W) - \beta W_{t+1}'(W)).$$

Rearranging, and using the fact that $W_0'(W) = 1$, we can write this as

$$\begin{aligned} & -\beta \left[\mathcal{H}'(\frac{W^*(H) - \beta W}{1-\beta}) - \mu\mathcal{H}'(\frac{W_0(W) - \beta W_1(W)}{1-\beta}) \right] \\ & - \sum_{t=1}^{\infty} (\mu\beta)^t \beta \left[\mathcal{H}'(\frac{W_{t-1}(W) - \beta W_t(W)}{1-\beta}) - \mu\mathcal{H}'(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) \right] W_t'(W). \end{aligned}$$

We know that for all $t \geq 1$, $W_t'(W) > 0$. In addition, we know $W^*(H) - \beta W < W_0(W) - \beta W_1(W)$ and that for all $t \geq 1$, $W_{t-1}(W) - \beta W_t(W) < W_t(W) - \beta W_{t+1}(W)$. Since $\mathcal{H}$ is concave, this implies that the above expression is negative. We conclude that the derivative is decreasing in W as required. $\blacksquare$

Given Fact A.2.7, we know from (33) that

$$W_n(H) = \arg\max_{W'} \left\{ (1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{u}{1-\beta}\right] + \mu\left[S\left(H\right) + \frac{W^*(H) - \beta W'}{H} + \beta V^*(W')\right]\right\}.$$

The derivative of the objective function is

$$\beta\left(\mu\frac{dV^*(W')}{dW} - \frac{1}{H}\right).$$

We know from (70) and (71) that for any $W' > \mathcal{W}(H^s)$ we have that

$$\mu\frac{dV^*(W')}{dW} < \mu\frac{dV^*(\mathcal{W}(H^s))}{dW} = \frac{1}{H^s} < \frac{1}{H}.$$

Accordingly, $W_n(H) \leq \mathcal{W}(H^s)$. Recall, however, that the value function may be kinked at $\mathcal{W}(H^s)$. It follows that $W_n(H)$ must either satisfy the first order condition

$$\mu\frac{dV^*(W_n(H))}{dW} = \frac{1}{H} \tag{83}$$

or equal $\mathcal{W}(H^s)$ if it is the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu\frac{dV^*(W)}{dW} \geq \frac{1}{H}.$$

11.2.2 The residents' problem

We are now ready to show that the policy rules defined by (25), (27), (30) solve the residents' problem given that the future price and continuation value are as described in (30) and (31) and (80). Our first observation is that the residents' policy choices are such that next period's wealth is at least as big as the threshold $\mathcal{W}(H')$.

Fact A.2.8. *Suppose that $H \geq H^s$ and let (g', W', H', P) solve problem (64) with future price and continuation value as described in (30) and (31) and (80). Then, $W' \geq W^*(H')$.*

Proof of Fact A.2.8. Suppose, to the contrary, that $W' < W^*(H')$. Then, from (30) and (31)

$$P(g', \beta\left(W' - cg'\right), H') = \mathcal{P}(H', W_n(H'), W')$$

and

$$V(g', \beta\left(W' - cg'\right), H') = V^*(W^*(H')) + \frac{W' - W^*(H')}{H'}.$$

Thus, the current price is

$$P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta \mathcal{P}(H', W_n(H'), W') - \underline{u}.$$

and the payoff is

$$(1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta\left(V^*(W^*(H')) + \frac{W' - W^*(H')}{H'}\right)\right]$$

Note that neither the price nor the payoff vary with respect to W' for any $W' < W^*(H')$. However, at $W' = W^*(H')$, the price jumps up reflecting the fact that the future price jumps from $\mathcal{P}(H', W_n(H'), W^*(H'))$ (which is less than C by Fact A.2.7) to C. If it is the case, that

$$(1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta C - \underline{u} \leq C$$

Then it is clear that we can replace (g', W', H', P) with a policy in which $W' = W^*(H')$ and we will have increased the value of the objective function. Since this is a contradiction, we can assume that

$$(1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta C - \underline{u} > C.$$

We can also write the payoff under (g', W', H', P) as

$$(1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta V^*(W^*(H'))\right]. \tag{84}$$

Now choose a wealth level $\widehat{W} > W^*(H')$ which keeps the current price equal to P, but makes the future price equal C. This price satisfies

$$(1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \widehat{W}}{H'} + \beta C - \underline{u} = P. \tag{85}$$

The policies $(g', \widehat{W}, H', P)$ yield a payoff

$$(1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \widehat{W}}{H'} + \beta V^*(\widehat{W})\right]. \tag{86}$$

Comparing (84) and (86), we see that choosing the alternative policies $(g', \widehat{W}, H', P)$ will increase the objective function if

$$V^*(\widehat{W}) - \frac{\widehat{W}}{H'} \geq V^*(W^*(H')) - \frac{W^*(H')}{H'}.$$

Given that V^* is strictly concave, this will be true if

$$\frac{dV^*(\widehat{W})}{dW} \geq \frac{1}{H'}.$$

We know from (83) that

$$\frac{dV^*(W_n(H'))}{dW} > \frac{1}{H'},$$

thus it is sufficient to show that $\widehat{W}$ is less than $W_n(H')$.

To establish this, note first from (85) that

$$\begin{aligned} & (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta \mathcal{P}(H', W_n(H'), W^*(H')) - \underline{u} \\ = \quad & (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \widehat{W}}{H'} + \beta C - \underline{u}. \end{aligned}$$

Cancelling terms and dividing through by β, this implies that

$$\mathcal{P}(H', W_n(H'), W^*(H')) + \frac{\widehat{W} - W^*(H')}{H'} = C.$$

Recall that, by definition,

$$\mathcal{P}(H', W_n(H'), W^*(H')) \equiv (1-H')\overline{\theta} + S(H') + \frac{W^*(H') - \beta W_n(H')}{H'} + \beta C - \underline{u}.$$

Thus, this implies that

$$(1-H')\overline{\theta} + S(H') + \frac{\widehat{W} - \beta W_n(H')}{H'} + \beta C - \underline{u} = C$$

or, equivalently, that

$$\mathcal{P}(H', W_n(H'), \widehat{W}) = C.$$

This equality implies that $\widehat{W} < W_n(H')$. Suppose, to the contrary, that $\widehat{W} \geq W_n(H')$. Then, by Fact A.2.7 and the fact that (by definition) $W^*(H') > \mathcal{W}(H')$, we have that

$$\begin{aligned} \mathcal{P}(H', W_n(H'), \widehat{W}) & \geq \mathcal{P}(H', W_n(H'), W_n(H')) \\ & > \mathcal{P}(H', W^*(H'), W^*(H')) \\ & > \mathcal{P}(H', \mathcal{W}(H'), \mathcal{W}(H')) = C, \end{aligned}$$

which is a contradiction. ■

Fact A.2.8 allows us to rewrite problem (64) as

$$\max_{(g',W',H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right)-\frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'}+\frac{W-\beta W'}{H'}+\beta V^{*}(W')\right] \\ s.t.\ g'\geq 0\ \&\ H'\geq H \\ P=(1-H')\overline{\theta}+B(\frac{g'/(1-\delta)}{(H')^{\alpha}})-\frac{c\left(\frac{g'}{1-\delta}-\beta g'\right)}{H'}+\frac{W-\beta W'}{H'}+\beta C-\underline{u} \\ P\leq C\ (=\text{if } H'>H) \\ W'\geq W^{*}(H') \end{array} \right\} \tag{87}$$

Our next result shows that public good provision is efficient.

Fact A.2.9. *Suppose that $H<H^{s}$ and let (g',W',H',P) solve problem (87) with $V^{*}(W')$ as described in (80). Then, $g'=(1-\delta)g^{o}(H')$.*

Proof of Fact A.2.9. The argument is identical to the proof of Fact A.2.4. ■

Fact A.2.9 allows us to write the residents' problem as

$$\max_{(W',H')} \left\{ \begin{array}{c} (1-\mu)\left[(1-H')\overline{\theta}+\beta C-\underline{u}+\frac{u}{1-\beta}\right]+S(H')+\frac{W-\beta W'}{H'}+\mu\beta V^{*}(W') \\ s.t.\ H'\geq H \\ (1-H')\overline{\theta}+S(H')+\frac{W-\beta W'}{H'}-(1-\beta)C\leq\underline{u}\ (=\text{if } H'>H) \\ W'\geq W^{*}(H') \end{array} \right\}. \tag{88}$$

Our next result ties down the optimal (W',H').

Fact A.2.10. *Suppose that $H\geq H^{s}$ and let (W',H') solve problem (88) with $V^{*}(W')$ as described in (80). Then,*

$$(W',H')=\left\{ \begin{array}{c} (W_{n}(H),H)\ \text{ if } W<W^{*}(H) \\ (W^{*}(H_{c}(W)),H_{c}(W))\ \text{ if } W\geq W^{*}(H) \end{array} \right. .$$

Proof of Fact A.2.10. There are two possibilities to consider: i) the price constraint holds with equality at the optimal policies, and ii) the price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). If

$$(1-H')\overline{\theta}+S(H')+\frac{W-\beta W'}{H'}-(1-\beta)C=\underline{u},$$

then $H'=\mathcal{H}((W-\beta W')/(1-\beta))$. The constraint that $H'\geq H$ then implies that $W-\beta W'\geq(1-\beta)\mathcal{W}(H)$ which means that

$$W'\leq\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}.$$

The constraint that $W' \geq W^*(H')$ implies that $W' \geq W^*(\mathcal{H}((W - \beta W')/(1-\beta)))$. Since

$$H_c(W) = \mathcal{H}(\frac{W - \beta W^*(H_c(W))}{1-\beta}),$$

this inequality requires that $W' \geq W^*(H_c(W))$. It follows that the range of feasible W' values is

$$W' \in [W^*(H_c(W)), \frac{W - (1-\beta)\mathcal{W}(H)}{\beta}].$$

For this interval to be non-empty, it is necessary that

$$\beta W^*(H_c(W)) + (1-\beta)\mathcal{W}(H) \leq W.$$

If this condition is not satisfied, then there exist no values of W' such that both $\mathcal{H}((W - \beta W')/(1-\beta)) \geq H$ and $W' \geq W^*(\mathcal{H}((W - \beta W')/(1-\beta)))$. This condition is equivalent to the requirement that $W \geq \beta W^*(H) + (1-\beta)\mathcal{W}(H)$. To see this, note that

$$\mathcal{P}(H, W^*(H), H_c^{-1}(H)) = C.$$

Thus,

$$H_c^{-1}(H) - \beta W^*(H) = (1-\beta)\mathcal{W}(H),$$

which implies that

$$H_c^{-1}(H) = \beta W^*(H) + (1-\beta)\mathcal{W}(H).$$

If $W \geq H_c^{-1}(H)$, then $H_c(W) \geq H$ and

$$\mathcal{P}(H_c(W), W^*(H_c(W)), W) = C.$$

This implies that

$$W - \beta W^*(H_c(W)) = (1-\beta)\mathcal{W}(H_c(W)) \geq (1-\beta)\mathcal{W}(H)$$

If $W < H_c^{-1}(H)$, then $H_c(W) < H$ and

$$\mathcal{P}(H_c(W), W^*(H_c(W)), W) = C.$$

This implies that

$$W - \beta W^*(H_c(W)) = (1-\beta)\mathcal{W}(H_c(W)) < (1-\beta)\mathcal{W}(H).$$

Using the fact that the price constraint binds, we can write the objective function as

$$C(1-\mu\beta) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W - \beta W'}{1-\beta}) - 1\right] + \mu\beta V^*(W')$$

Note from (80) that the form of the function $V^*(W')$ depends on whether W' is greater than or less than $\mathcal{W}(H^s)$. This requires us to distinguish three cases: a) $W \leq \beta\mathcal{W}(H^s) + (1-\beta)\mathcal{W}(H)$; b) $W \geq \mathcal{W}(H^s)$; and c) $W \in (\beta\mathcal{W}(H^s) + (1-\beta)\mathcal{W}(H), \mathcal{W}(H^s))$.

In case a), $\frac{W-(1-\beta)\mathcal{W}(H)}{\beta} \leq \mathcal{W}(H^s)$. Using (80), the optimal W' must therefore solve the problem

$$\max_{W'} \left\{ \begin{array}{c} \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta}) - 1\right] + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta}) - 1\right] \\ s.t.\ W' \in [W^*(H_c(W)), \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}] \end{array} \right\}.$$

We claim that the objective function is decreasing in W' and hence that the solution to this problem is $W^*(H_c(W))$. Ignoring multiplicative constants, the derivative of the objective function is

$$-\mathcal{H}'(\frac{W-\beta W'}{1-\beta})\beta + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mathcal{H}'(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta})(W_t'(W') - \beta W_{t+1}'(W')).$$

Rearranging, and using the fact that $W_0'(W') = 1$, we can write this as

$$\begin{aligned} &-\beta\left[\mathcal{H}'(\frac{W-\beta W'}{1-\beta}) - \mu\mathcal{H}'(\frac{W_0(W')-\beta W_1(W')}{1-\beta})\right] \\ &-\sum_{t=1}^{\infty} (\mu\beta)^t \beta\left[\mathcal{H}'(\frac{W_{t-1}(W')-\beta W_t(W')}{1-\beta}) - \mu\mathcal{H}'(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta})\right] W_t'(W'). \end{aligned}$$

We know that for all $t \geq 1$, $W_t'(W') > 0$. In addition, we know $W - \beta W' < W_0(W') - \beta W_1(W')$ and that for all $t \geq 1$, $W_{t-1}(W') - \beta W_t(W') < W_t(W') - \beta W_{t+1}(W')$. Since $\mathcal{H}$ is concave, this implies that the above expression is negative.

In case b), $H_c(W) = \mathcal{H}(W) \geq H^s$ and $W^*(H_c(W)) = W \geq \mathcal{W}(H^s)$. Using (80), the optimal W' must solve the problem

$$\max_{W'} \left\{ \begin{array}{c} \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta}) - 1\right] + \left(\frac{\mu\beta}{1-\mu\beta}\right)\mu\overline{\theta}(\mathcal{H}(W') - 1) \\ s.t.\ W' \in [W, \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}] \end{array} \right\}.$$

Ignoring multiplicative constants, the derivative of the objective function is

$$\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W') - \frac{\beta}{1-\beta}\mathcal{H}'(\frac{W-\beta W'}{1-\beta}).$$

The concavity of the function $\mathcal{H}(W)$, implies that this derivative is negative for all $W' \in [W, \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]$. The optimal choice of wealth is therefore $W = W^*(H_c(W))$.

In case c), the optimal choice of wealth maximizes the objective function:

$$\left\{ \begin{array}{c} \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta}) - 1\right] + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta}) - 1\right] \quad \text{if } W' \in [W^*(H_c(W)), \mathcal{W}(H^s)) \\ \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta}) - 1\right] + \left(\frac{\mu\beta}{1-\mu\beta}\right)\mu\overline{\theta}(\mathcal{H}(W') - 1) \quad \text{if } W' \in [\mathcal{W}(H^s), \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}] \end{array} \right.$$

As shown in case a), this objective function is decreasing on $[W^*(H_c(W)), \mathcal{W}(H^s))$, and, as shown in case b), it is also decreasing on $[\mathcal{W}(H^s), \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]$. Given that the objective function is continuous at $\mathcal{W}(H^s)$, the optimal choice of wealth is $W^*(H_c(W))$.

We conclude that if the price constraint holds with equality at the optimal policies, then the optimal policies are $(W^*(H_c(W)), H_c(W))$. A necessary condition for this to be the solution is that $W \geq \beta W^*(H) + (1-\beta)\mathcal{W}(H)$. Note for future reference that the payoff from this candidate solution is $V^*(W)$.

Possibility ii). If the price constraint holds as an inequality at the optimal policies, then $H' = H$ and $W - \beta W' < (1-\beta)\mathcal{W}(H')$. This means that

$$W' > \frac{W - (1-\beta)\mathcal{W}(H)}{\beta}.$$

The constraint that $W' \geq W^*(H')$ requires that $W' \geq W^*(H)$. The optimal choice of wealth therefore solves the problem

$$\max_{W'} \left\{ \begin{array}{c} (1-\mu)\left[(1-H)\overline{\theta} + \beta C - \underline{u} + \frac{\underline{u}}{1-\beta}\right] + S(H) + \frac{W-\beta W'}{H} + \mu\beta V^*(W') \\ W' \geq W^*(H) \end{array} \right\}$$

We claim that the solution is $W_n(H)$ as defined in (33). Notice from (29) that the objective function in this problem is equal to

$$(1-\mu)\left[\mathcal{P}(H, W', W) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W-\beta W'}{H} + \beta V^*(W')\right].$$

Given that W enters as an additive constant and can have no impact on the solution, this is equivalent to an objective function

$$(1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W^*(H)-\beta W'}{H} + \beta V^*(W')\right].$$

From (33) and Fact A.2.7, we know that

$$W_n(H) = \arg\max_{W'}(1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W^*(H)-\beta W'}{H} + \beta V^*(W')\right]$$

and that $W_n(H) > W^*(H)$. Thus, the solution is $W_n(H)$ as claimed.

We conclude that if the price constraint holds as an inequality at the optimal policies, then the optimal policies are $(W_n(H), H)$. A necessary condition for this to be the solution is that $W < \beta W_n(H) + (1-\beta)\mathcal{W}(H)$. If this condition is not satisfied, then the price constraint cannot hold as an inequality when the policies are $(W_n(H), H)$. The payoff from this candidate solution is

$$(1-\mu)\left[\mathcal{P}(H,W_n(H),W)+\frac{\underline{u}}{1-\beta}\right]+\mu\left[S(H)+\frac{W-\beta W_n(H)}{H}+\beta V^*(W_n(H))\right]$$

Which possibility arises? Having understood the two possibilities, we can now analyze which one arises. A necessary condition for possibility ii) to be the solution is that $W < \beta W_n(H) + (1-\beta)\mathcal{W}(H)$. Furthermore, a necessary condition for possibility i) to be the solution is that $W \geq \beta W^*(H) + (1-\beta)\mathcal{W}(H)$. Given that by Fact A.2.7, $W_n(H) > W^*(H)$, we can conclude that the solution is $(W_n(H), H)$ if $W < \beta W^*(H) + (1-\beta)\mathcal{W}(H)$ and $(W^*(H_c(W)), H_c(W))$ if $W \geq \beta W_n(H)+(1-\beta)\mathcal{W}(H)$. For values of W in the interval $[\beta W^*(H)+(1-\beta)\mathcal{W}(H), \beta W_n(H)+(1-\beta)\mathcal{W}(H))$ both possibilities are feasible. Thus, which possibility is optimal depends on a comparison of the payoffs. We claim that:

$$(W', H') = \begin{cases} (W_n(H), H) \text{ if } W < W^*(H) \\ (W^*(H_c(W)), H_c(W)) \text{ if } W \geq W^*(H) \end{cases}.$$

Define the function $\varphi(W;H)$ on the interval $[\beta W^*(H)+(1-\beta)\mathcal{W}(H), \beta W_n(H)+(1-\beta)\mathcal{W}(H))$ to be equal to the difference between the payoffs from the two candidate solutions; that is,

$$\varphi(W;H) = V^*(W)-(1-\mu)\left[\mathcal{P}(H,W_n(H),W)+\frac{\underline{u}}{1-\beta}\right]-\mu\left[S(H)+\frac{W-\beta W_n(H)}{H}+\beta V^*(W_n(H))\right].$$

Condition (34) implies that $\varphi(W^*(H);H) = 0$. Thus, it is sufficient to show that $\varphi(W;H)$ is increasing in W. Differentiating, we have that

$$\frac{d\varphi(W;H)}{dW} = \frac{dV^*(W)}{dW} - \frac{1}{H}.$$

We know that $W < W_n(H)$ and from (83) that

$$\frac{dV^*(W_n(H))}{dW} > \frac{1}{H}.$$

Thus, it follows from the concavity of $V^*(W)$, that $\varphi(W;H)$ is increasing in W as required. ■

Using Facts A.2.9 and A.2.10, it is clear that the residents want to follow the equilibrium policy rules described in (25), (27), and (30).

11.2.3 Verifying form of value function

It remains to verify that the value function as described by (31) and (80) satisfies (18). Suppose first that (g, b, H) is such that $W \geq W^*(H)$. Then, we have that

$$\begin{aligned}
& (1-\mu)\left[P(\cdot)+\frac{\underline{u}}{1-\beta}\right]+\mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right)-T(\cdot)+\beta V(g'(\cdot), b'(\cdot), H'(\cdot))\right] \\
= \; & (1-\mu)\left[C+\frac{\underline{u}}{1-\beta}\right]+\mu\left[S(H_c(W))+\frac{W-\beta W^*(H_c(W))}{H_c(W)}+\beta V^*(W^*(H_c(W)))\right] \\
= \; & V^*(W),
\end{aligned}$$

as required. Now suppose that (g, b, H) is such that $W < W^*(H)$. Then, using (34), we have that

$$\begin{aligned}
& (1-\mu)\left[P(\cdot)+\frac{\underline{u}}{1-\beta}\right]+\mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right)-T(\cdot)+\beta V(g'(\cdot), b'(\cdot), H'(\cdot))\right] \\
= \; & (1-\mu)\left[\mathcal{P}(H, W_n(H), W)+\frac{\underline{u}}{1-\beta}\right]+\mu\left[S(H)+\frac{W-\beta W_n(H)}{H}+\beta V^*(W_n(H))\right] \\
= \; & (1-\mu)\left[\mathcal{P}(H, W_n(H), W^*(H))+\frac{\underline{u}}{1-\beta}\right]+\mu\left[S(H)+\frac{W^*(H)-\beta W_n(H)}{H}+\beta V^*(W_n(H))\right] \\
& +\frac{W-W^*(H)}{H} \\
= \; & V^*(W^*(H))+\frac{W-W^*(H)}{H},
\end{aligned}$$

as required. ■

12 Appendix 3: Proofs of Propositions 5, 6, and 7

12.1 Proof of Proposition 5

We begin by verifying the claims about housing, public good investment, and wealth. We first claim that W_0 exceeds $W^*(H_0)$. Remember that $W^*(H)$ is increasing. Our Assumptions imply that $H_0 < H^s \leq \mathcal{H}(W_0)$, so it is the case that $W^*(H_0) < W^*(H^s) \leq W^*(\mathcal{H}(W_0))$. Recall also that $W^*(H)$ is equal to $\mathcal{W}(H)$ for housing levels larger than H^s which implies that $W^*(\mathcal{H}(W_0)) = \mathcal{W}(\mathcal{H}(W_0))$. Given that the function $\mathcal{H}(W)$ is the inverse of the function $\mathcal{W}(H)$, it follows that $W^*(\mathcal{H}(W_0)) = W_0$.

Given that W_0 exceeds $W^*(H_0)$, it follows from (25) that $(g_1, H_1) = ((1-\delta)g^o(H_c(W_0)), H_c(W_0))$. Next note that $H_c(W_0) = \mathcal{H}(W_0)$. As shown in the proof of the Theorem (Fact A.2.1), if $W \geq W^*(H)$ and $H \geq H^s$, then $H_c(W) = \mathcal{H}(W)$. But we know that $W_0 \geq W^*(H^s)$ and hence $H_c(W_0) = \mathcal{H}(W_0)$. Given this, we have that $(g_1, H_1) = ((1-\delta)g^o(\mathcal{H}(W_0)), \mathcal{H}(W_0))$ and, from (27), that

$$W_1 = cg_1 - (1+\rho)b_1 = cg_1 - (1+\rho)\left(\frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W^*(\mathcal{H}(W_0))}{1+\rho}\right) = W_0.$$

This means that $(W_1, H_1) = (W_0, \mathcal{H}(W_0))$.

Since $W^*(\mathcal{H}(W_0)) = W_0$, it follows from (25) that $(g_2, H_2) = ((1-\delta)g^o(H_c(W_0)), H_c(W_0))$. As just argued, we have that $H_c(W_0) = \mathcal{H}(W_0)$ and thus $(g_2, H_2) = ((1-\delta)g^o(\mathcal{H}(W_0)), \mathcal{H}(W_0))$ which in turn implies that $W_2 = W_0$. Repeated application of this argument implies that for all $t \geq 1$, $(W_t, H_t, g_t) = (W_0, \mathcal{H}(W_0), (1-\delta)g^o(\mathcal{H}(W_0)))$.

Turning to what is happening to debt, from (27), we have that

$$\begin{aligned} b_1 - b_0 &= \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho} - \frac{cg_0 - W_0}{1+\rho} \\ &= \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - cg_0}{1+\rho}. \end{aligned}$$

Furthermore, for all $t \geq 2$

$$b_{t+1} - b_t = \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho} - \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho} = 0.$$

Thus, the value of outstanding debt increases by $c\left((1-\delta)g^o(\mathcal{H}(W_0)) - g_0\right)$ in period 0 and then remains constant. Finally, given that for all periods $W_t \geq W^*(H_t)$, (30) implies that the price of housing is constant at the construction cost C. $\blacksquare$

12.2 Proof of Proposition 6

Recall first that our Assumptions imply that $H_0 < H^s$. Since $W_0 \geq W^*(H_0)$, (25) and (27) tell us that $H_1 = H_c(W_0)$ and $W_1 = W^*(H_1)$. Repeated application of (25) and (27) reveals that in each period t beyond period 1, H_t is equal to $H_c(W^*(H_{t-1}))$ and W_t is equal to $W^*(H_t)$. As shown in the proof of the Theorem (Fact A.2.6), the sequence of housing levels $\langle H_t \rangle_{t=0}^{\infty}$ is increasing and converges asymptotically to H^s. This implies that new construction takes place in each period and that the size of the community approaches H^s. Given that the threshold wealth function $W^*(\cdot)$ is increasing and $W^*(H^s) = \mathcal{W}(H^s)$, this implies that wealth is increasing and converging asymptotically to $\mathcal{W}(H^s)$. Equation (25) tells us that in each period $t \geq 1$, $g_t = (1-\delta)g^o(H_{t+1})$ implying that in each period t the public good level enjoyed by the residents is $g^o(H_{t+1})$.

Regarding the value of outstanding debt, equation (27) implies that for all t

$$b_{t+1} - b_t = \frac{cg_{t+1} - W_{t+1}}{1+\rho} - \frac{cg_t - W_t}{1+\rho}.$$

Thus, since $W_{t+1} > W_t$, we have that

$$\begin{aligned}(1+\rho)(b_{t+1} - b_t) &= c(g_{t+1} - g_t) - (W_{t+1} - W_t) \\ &< c(g_{t+1} - g_t),\end{aligned}$$

as required. Equation (30) implies that the price of housing is equal to the construction cost in each period. ■

12.3 Proof of Proposition 7

Since $W_0 < W^*(H_0)$, (25) and (27) tell us that $H_1 = H_0$ and $W_1 = W_n(H_0)$. As shown in the proof of the Theorem (Fact A.2.7), the fact that $H_0 < H^s$, implies that $W_n(H_0) > W^*(H_0)$ and that $P(H_0, W_n(H_0), W_0) < C$. If $W_n(H_0) < \mathcal{W}(H^s)$, we can set $W_0 = W_n(H_0)$ and reapply the arguments made to establish Proposition 6. If $W_n(H_0) = \mathcal{W}(H^s)$, then we can reapply the arguments made in the proof of Proposition 5. ■

13 Appendix 4: Numerical Analysis

In this Appendix we investigate the existence of equilibrium in our model using numerical methods. We first describe our parametrization strategy, and in particular, the set of parameters over which our investigation is conducted. We then describe our method for checking Assumptions 1-2, constructing equilibrium objects, and analyzing whether these objects have the required properties.

13.0.1 Parameters

To start we need to specify the public good benefit function. We assume that it is a power function: $B(x) = B_0 x^\sigma/\sigma$, where σ is positive but less than one. With this assumption, there are ten parameters in our model:

β	δ	c	$\underline{u}$	$\bar{\theta}$	μ	σ	α	B_0	C
$\frac{1}{1.06}$	0.1	1	0	1	$(0,1)$	$(0,1)$	$[0,1]$	?	?

Parameters β and δ are commonly used in the literature and we set them accordingly. In particular, we set the discount rate β equal to $1/1.06$ and the depreciation rate of the public good δ equal to 0.1.

Parameters c, $\underline{u}$, and $\bar{\theta}$ can be normalized. We set them equal to 1, 0, and 1, respectively. To see that the cost of the public good c can be normalized note that the optimized public good surplus function defined in (5) can be written as $S(H) = s_0 \cdot H^{s_1}$ where $s_0 = \frac{1-\sigma}{\sigma} B_0 \left[\frac{B_0}{c[1-\beta(1-\delta)]}\right]^{\frac{\sigma}{1-\sigma}}$ and $s_1 = (1-\alpha)\frac{\sigma}{1-\sigma}$. Thus, while c influences the level of surplus obtained from any given housing stock, any change in c can be mimicked by an appropriate change in the benefit function parameter B_0. The utility households obtain from living outside the community $\underline{u}$ can be normalized, because it does not play an independent role. It is both intuitive and straightforward to verify that what matters is $\underline{u} + (1-\beta)C$. Thus, any change in $\underline{u}$ can be mimicked by an appropriate change in the housing price C. Finally, to see that the upper bound on the preference distribution $\bar{\theta}$ can be normalized, take a given parameterization incorporating the normalizations to parameters c and $\underline{u}$. Consider now a different parameterization that keeps all parameters unchanged except $\bar{\theta}$ is set to one, B_0 is adjusted so that s_0 is divided by $\bar{\theta}$, and C is divided by $\bar{\theta}$. Such a parameterization will generate a model that is isomorphic to the original model: all value functions, prices and wealth levels will equal their respective original values divided by $\bar{\theta}$.

This leaves five parameters: μ, σ, α, B_0, and C. The first three range between 0 and 1 and have simple interpretations: μ reflects population turnover; σ the concavity of public good benefits;

and α the congestibility of the public good. Accordingly, it is easy to specify a priori a range of plausible values for these parameters. This is not the case for the remaining two parameters B_0 and C. The way we deal with this pair is to note that, given the rest of the parameters, they determine the values of H^s and H^o. Recall that H^o is the socially optimal housing level and H^s is the steady state level under the wealth accumulation path. We know that both these housing levels lie between 0 and 1 and that H^s is less than H^o. Accordingly, our approach is to specify H^s and H^o directly along with the other parameter values and then check to see whether there are underlying values B_0 and C which generate these housing levels.

In sum, we have five fixed parameters $(\beta, \delta, c, \underline{u}, \bar{\theta})$ and five parameters which vary $(\mu, \sigma, \alpha, B_0, C)$. For the variable parameters, we let μ take ten values: from 0.90 to 0.99 with an increment of 0.01; σ take nine values: from 0.1 to 0.9 with an increment of 0.1; and α take eleven values from 0 to 1 with an increment of 0.1. Finally, for a given set of values for μ, σ, and α, we entertain eighty one possible pairs of (B_0, C) that correspond to eighty one pairs of (H^s, H^o). To get these eighty one pairs, we allow H^o to vary from 0.1 to 0.9 with an increment of 0.1 and, for each such H^o, we consider nine values of H^s of the form $H^s = rH^o$, where r varies from 0.1 to 0.9.

In total, we have $10 \cdot 11 \cdot 9 \cdot 9 \cdot 9 = 80190$ different parameterizations. For each of these parameterizations, we implement the numerical procedure described below. We find that:

1. In 37.4% of the cases there does not exist a pair of parameters (B_0, C) in the feasible range (i.e., $\Re^2_+$) that, given the other parameters, generate the specified values of H^s and H^o.

2. In 37.6% of the cases there does not exist an H_0 for which Assumptions 1-2 are satisfied.

3. In 24.8% (19854) of the cases there exists a non-empty interval $[\underline{H}, H^s)$ such that Assumptions 1-2 are satisfied for any $H_0 \in [\underline{H}, H^s)$. Moreover, for all $H_0 \in [\underline{H}, H^s)$, there exists an equilibrium threshold wealth function.

4. In 1.3% (1024) of the cases there exists a non-empty interval $[\underline{H}, H^s)$ such that Assumptions 1-2 are satisfied for any $H_0 \in [\underline{H}, H^s)$. Moreover, there exists $\hat{H} \in (\underline{H}, H^s)$ such that for all $H_0 \in [\hat{H}, H^s)$, there exists an equilibrium threshold wealth function.

Summarizing these findings, we have 20878 different parameterizations under which there exist an interval of initial housing levels H_0 that satisfy Assumptions 1-2. These are the relevant parameterizations for our purposes. The key point to note is that for 19854 of these, there exists an equilibrium threshold wealth function for *any* of the initial housing levels H_0 that satisfy Assumptions 1-2. Thus, for over 95% of the parameterizations satisfying our assumptions, existence of an equilibrium threshold wealth function is not problematic. In the remaining 1024 cases, there exists an equilibrium threshold wealth function for only *some* of the initial housing levels H_0 that satisfy Assumptions 1-2. For a range of initial housing levels, an equilibrium threshold wealth function does not exist.

The overwhelming majority of these problematic cases feature high μ. In fact, 880 of the 1024 cases have $\mu = 0.99$.[37] We have investigated what is going on in a handful of randomly picked cases in this category. In all these cases, there does exist an equilibrium. Moreover, it involves a threshold wealth function W^* and is described by $\mathcal{E}(W^*)$. However, the threshold wealth function does not belong to the set Ψ and, moreover, the associated function $V^*(W)$ is not concave.[38]

Figure 2 illustrates one of these problematic cases. The associated values of the parameters (μ, σ, α) are $(0.99, 0.5, 1)$. The values of (B_0, C) are chosen to generate a (H^s, H^o) pair equal to $(0.05, 0.1)$. Panel A illustrates the threshold wealth function $W^*(H)$ along with the function $\mathcal{W}(H)$. Notice that $W^*(H)$ is kinked at housing level 0.037 and is not differentiable. Moreover, as illustrated, at housing level 0.037, $W^*(H)$ is equal to $\mathcal{W}(H)$, as opposed to exceeding it. Panel B illustrates the associated $V^*(W)$ function. Between wealth levels 0.0036 and 0.008, the function is not concave. Panel C illustrates the function $W_n(H)$ along with the function $W^*(H)$. Notice that at housing level 0.037, $W_n(H)$ is equal to $W^*(H)$ and hence equal to $\mathcal{W}(H)$.

What is happening in this equilibrium is that at housing level 0.037, if the community is endowed with wealth $\mathcal{W}(H)$, the residents are not willing to build wealth to attract new residents. This is the case despite the fact that the housing level is less than H^s. Recall that, when we introduced it in Section 6.2, we stated that H^s was the smallest housing level such that, if the community is endowed with wealth $\mathcal{W}(H)$, it will never choose to build wealth to attract new residents. However, this was under the assumption that increasing wealth by ϵ will simply increase the future housing level to $\mathcal{H}(\mathcal{W}(H)+\epsilon)$ and future wealth to $\mathcal{W}(H)+\epsilon$. At housing level 0.037, the future consequences of increasing wealth marginally are much more complicated and determined by the behavior of the threshold wealth function $W^*(H)$ between 0.037 and H^s. Evidently, these consequences are less advantageous to current residents than simply increasing the future housing level to $\mathcal{H}(\mathcal{W}(H) + \epsilon)$ and future wealth to $\mathcal{W}(H) + \epsilon$. The value of these changes is reflected in

[37] Another noteworthy pattern is that the problematic cases happen mostly for values of r (the ratio of H^s to H^o) in the medium-high range. There are 444 cases with $r = 0.7$; 182 cases with $r = 0.6$; and 105 cases with $r = 0.5$. However, there are only 48 cases for $r = 0.8$. Exactly why r matters is unclear.

[38] Because the concavity of $V^*(W)$ is lost, we cannot use the Theorem to conclude that the candidate equilibrium $\mathcal{E}(W^*)$ associated with W^* is indeed an equilibrium. Instead, we rely on numerical methods to confirm that $\mathcal{E}(W^*)$ is indeed an equilbrium.

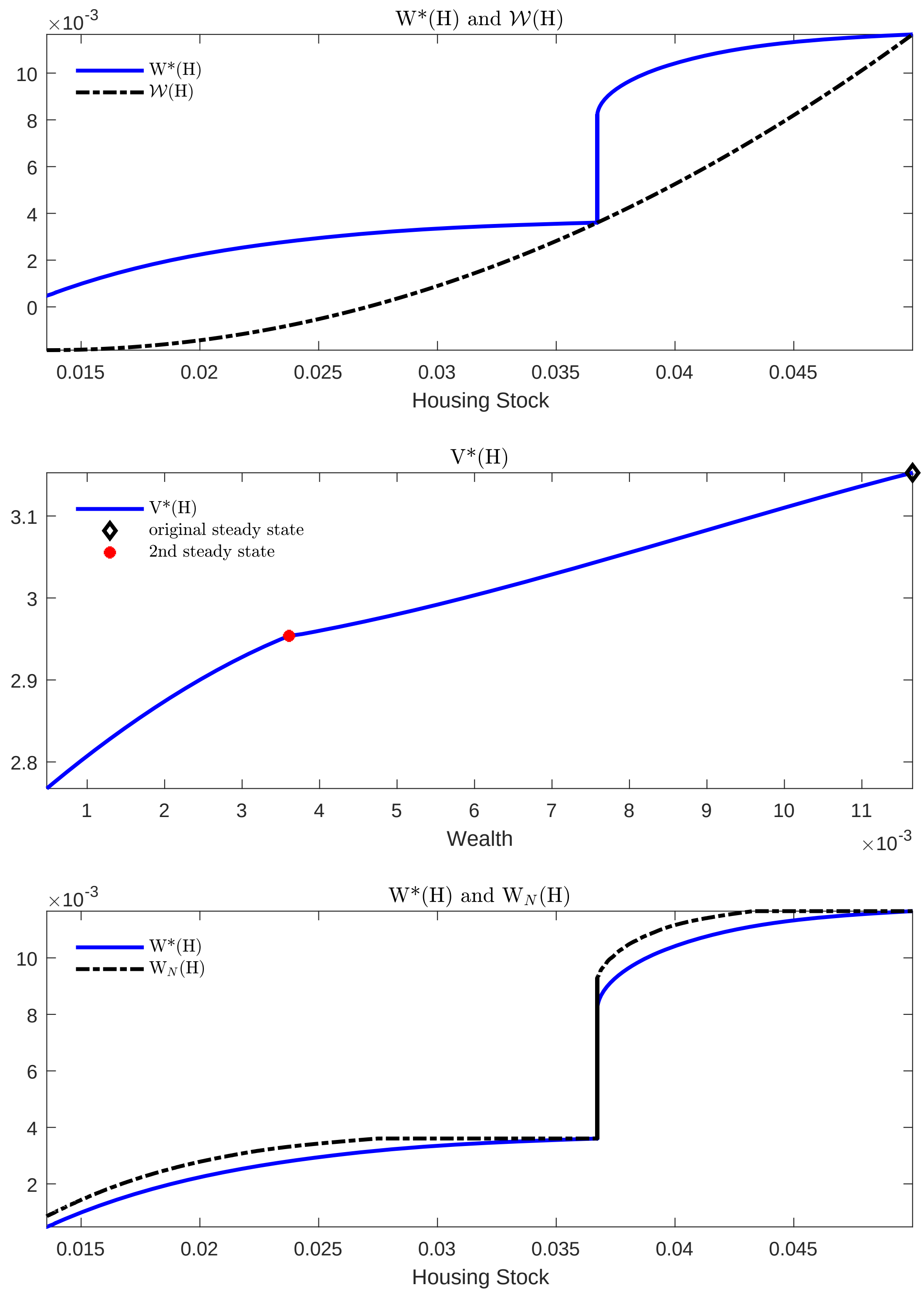

Figure 2: A problematic case

the derivative of the value function at wealth level $\mathcal{W}(0.037)$. At the kink, it must be the case that $\lim_{W \searrow \mathcal{W}(0.037)} \mu dV^*(W)/dW$ is less than $1/0.037$.

Substantively, the evolution of the community in this equilibrium differs from that described in Propositions 6 and 7. If the community's initial wealth W_0 is less than $\mathcal{W}(0.037)$ it will converge asymptotically to $(\mathcal{W}(0.037), 0.037)$ as opposed to $(\mathcal{W}(H^s), H^s)$. Qualitatively, the way in which the community develops remains the same, it is just that it approaches a different point. This point involves less development, so the extent to which the community will be undersized is enhanced. It should be noted, however, that the steady state $(\mathcal{W}(0.037), 0.037)$ is not stable: any positive wealth shock will result in the community converging to $(\mathcal{W}(H^s), H^s)$.

13.1 Numerical Procedure

For a given set of parameters values we take a 100000 point uniform grid over [0, 1]. We find the smallest point $\underline{H}$ at which Assumptions 1 and 2 are satisfied. If such a point does not exist, we stop. Otherwise, we search for the smallest H_0 greater or equal to $\underline{H}$ for which an equilibrium threshold wealth function can be shown to exist. If this smallest H_0 is equal to $\underline{H}$ then, for these parameter values, an equilibrium threshold wealth function exists whenever the Assumptions are satisfied.

13.1.1 Constructing equilibrium

Showing that an equilibrium threshold wealth function exists for a given set of parameter values and initial conditions (H_0, W_0) amounts to showing that we can construct an increasing function $W^*(H)$ on $[H_0, H^s]$ such that: i) $W^*(H) > \mathcal{W}(H)$ for all $H \in [H_0, H^s)$; ii) $W^*(H^s) = W(H^s)$; iii) the resulting $V^*(W)$ is increasing and concave; and iv) the indifference condition is satisfied. Two notes are in order. Rather than constructing function $W^*(H)$ directly it is more convenient to obtain it indirectly by constructing its inverse function, $H^*(W)$. This naturally implies that lower bound on W's, $\underline{W}$, for which the function $H^*(W)$ should be constructed must be such that $H^*(\underline{W}) \leq H_0$.

We assume and ex-post confirm that $V^*(W)$ is kinked at $\mathcal{W}(H^s)$. This implies, as shown in Fact A.4.1 below, that $W_n(H) = \mathcal{W}(H^s)$ on $[H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}, H^s]$. We use this fact in the first leg of our procedure.

Leg 1. We start by constructing the equilibrium objects in the neighborhood of the steady state $(\mathcal{W}(H^s), H^s)$. We rely on knowledge of $\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW}$ (established in the proof of

Fact A.4.1 below) to construct an approximation to the function $V^*(W)$ in this neighborhood. In particular:

1. For H close to H^s, $W = W^*(H)$ should be close to $\mathcal{W}(H^s)$. Consider interval $[W_I, \mathcal{W}(H^s)]$, where $W_I = \mathcal{W}(H^s)(1-\varepsilon)$ and ε is set to be small: specifically, $\varepsilon = 5 \cdot 10^{-4}$.

2. Take a twenty one point uniform grid on $[W_I, \mathcal{W}(H^s)]$. Call this grid $\vec{W}_I$. For each point on the grid, approximate function $V^*(W)$ using its first order Taylor expansion at $\mathcal{W}(H^s)$:

$$V^*(W) = V^*(\mathcal{W}(H^s)) + \lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW}(W - \mathcal{W}(H^s)),$$

where $\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW}$ is as computed in the proof of Fact A.4.1 below.

3. For each $W \in \vec{W}_I$ construct $H^*(W)$ as the solution to the following indifference condition:

$$V^*(W) = \mathcal{P}(H^*(W), \mathcal{W}(H^s), W) + \mu\overline{\theta}(H^*(W) - 1) + \mu\beta(V^*(\mathcal{W}(H^s)) - C).$$

If for any W, $H^*(W)$ is lower than $H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}$, reset ε to a smaller number, e.g. $\varepsilon = \varepsilon/2$, and repeat Steps 2-3.

4.

- Verify that $H^*(W)$ is increasing, that $H^*(W) > \mathcal{H}(W)$, and that $V^*(W)$ is increasing and concave on $[W_I, \mathcal{W}(H^s)]$. To do this, check that for any point A on $\vec{W}_I$

$$(1+\epsilon)H^*(A) > \mathcal{H}(A);$$

check that for any two adjacent points on $\vec{W}_I$, $A < B$,

$$(1+\epsilon)H^*(B) \geq H^*(A) \text{ and } (1+\epsilon)V^*(B) \geq V^*(A);$$

and check that for any three adjacent points on $\vec{W}_I$, $A < B < C$,

$$(1+\epsilon)V^*(B) \geq \alpha V^*(A) + (1-\alpha)V^*(C),$$

where $\alpha = \frac{B-A}{C-A}$ and $\epsilon = 10^{-4}$. A small ϵ is necessary to avoid interpreting numerical error in evaluating V when (a) it is near linear and (b) grid points are too close to each other, as a failure of concavity. We now have constructed $H^*(W)$ on the interval $[W_I, \mathcal{W}(H^s)]$. Observe that, by construction, we have that $H^*(\mathcal{W}(H^s)) = H^s$ and that the indifference condition (34) is satisfied for all W's on the grid.

- If not, find $\underline{W} \in [W_I, W^s]$ such that the four conditions above are satisfied on $[\underline{W}, \mathcal{W}(H^s)]$ and stop. Equilibrium exists for all $H_0 \geq \max(H^*(\underline{W}), \underline{H})$.

It is important to note here that we have built the equilibrium objects only on our grid. In what follows, when necessary, to construct either H^* or V^* at any other point on $[W_I, W(H^s)]$ we will use a shape preserving (cubic Hermite) interpolant of the relevant function.

Leg 2. We now extend our construct to the left using the knowledge of $H^*(W)$ and $V^*(W)$ on $\vec{W}_I$. The idea is to build $H^*(W)$ and $V^*(W)$ at wealth levels for which we know $H_c(W)$ and $W^*(H_c(W))$. Clearly, the lowest value of wealth for which this can be done is the wealth level for which $H_c(W) = H^*(W_I)$. Call that point W_{II}. Consider now interval $[W_{II}, W_I]$. By construction, we have that

$$\mathcal{P}(H^*(W_I), W_I, W_{II}) = C.$$

1. For each $W_1 \in \vec{W}_I \backslash \mathcal{W}(H^s)$ find $W_2 = W_2(W_1)$ (this is an abuse of notation, but we do so to avoid adding more notation) such that

$$\mathcal{P}(H^*(W_1), W_1, W_2) = C.$$

Note that $W_{II} = W_2(W_I)$. Call this collection of points $\vec{W}_{II}$. If all elements of $\vec{W}_{II} < W_I$ proceed to the next step. Otherwise, return to Leg 1, set $\varepsilon = \varepsilon/2$ and proceed.

2. For each $W_2 \in \vec{W}_{II}$ construct function $V^*(W)$ as follows:

$$V^*(W_2) = C + \mu\theta(H^*(W_1) - 1) + \mu\beta(V^*(W_1) - C).$$

where W_1 is such that $W_2 = W_2(W_1)$.

3. For each $W_2 \in \vec{W}_{II}$ construct function $H^*(W)$ as the solution to the following indifference condition:

$$V^*(W_2) = \max_{W_n > W_1} \left\{ \begin{array}{c} \mathcal{P}(H, W_n, W_2)) + \mu\overline{\theta}(H-1) + \mu\beta(\tilde{V}(W_n) - C) \\ s.t.\ \mathcal{P}(H, W_n, W_2) \leq C \end{array} \right\}$$

where W_1 is such that $W_2 = W_2(W_1)$ and $\tilde{V}(W_n)$ is a shape preserving (cubic Hermite) interpolant of $V^*(W)$ on $[W_I, \mathcal{W}(H^s)]$.[39]

The indifference condition above differs from the one in (34) because of the constraint that $W_n > W_1$. This is without loss of generality because, as shown in Fact A.4.3 below, all wealth levels W_n, for which the inequality $\mathcal{P}(H, W_n, W_2) \leq C$ holds, will necessarily be larger than W_1.

[39] Again, we use interpolant here because we have only constructed V^* at the grid points, but the search for W_n is over the entire interval $(W_1, \mathcal{W}(H^s)]$.

Indeed, Fact A.4.3 states that $W^*(H_c(W^*(H))) < W_n(H)$, and hence that $W_1 = W^*(H_c(W_2))$ should be less than $W_n(H^*(W_2))$.

4.

- Verify that $H^*(W)$ is increasing, that $H^*(W) > \mathcal{H}(W)$, and that $V(W)$ is increasing and concave on $[W_{II}, \mathcal{W}(H^s)]$. To do this, check that for any point A on $\vec{W}_{II} \cup \vec{W}_I$

$$(1+\epsilon)H^*(A) > \mathcal{H}(A);$$

check that for any two adjacent points on $\vec{W}_{II} \cup \vec{W}_I$, $A < B$,

$$(1+\epsilon)H^*(B) \geq H^*(A) \text{ and } (1+\epsilon)V^*(B) \geq V^*(A);$$

and check that for any three adjacent points on $\vec{W}_{II} \cup \vec{W}_I$, $A < B < C$,

$$(1+\epsilon)V^*(B) \geq \alpha V^*(A) + (1-\alpha)V^*(C),$$

where $\alpha = \frac{B-A}{C-A}$ and $\epsilon = 10^{-4}$.

Note that by construction we have that $H^*(\mathcal{W}(H^s)) = H^s$ and that the indifference condition (34) is satisfied for all W's on the grid.

- If not, find $\underline{W} \in [W_{II}, W_I]$ such that the four conditions above are satisfied on $[\underline{W}, W^s]$ and stop. Equilibrium exists for all $H_0 \geq \max(H^*(\underline{W}), \underline{H})$.

Leg 3. We now extend our construct to the left using the knowledge of $H^*(W)$ and $V^*(W)$ on $\vec{W}_{II} \cup \vec{W}_I$. The idea again is to build $H^*(W)$ and $V^*(W)$ at wealth levels for which we know $H_c(W)$ and $W_c(W)$. Clearly, the lowest value of wealth for which this can be done is the wealth level for which $H_c(W) = H^*(W_{II})$. Call that point W_{III}. By construction, we have that

$$\mathcal{P}(H^*(W_{II}), W_{II}, W_{III}) = C.$$

1. For each $W_2 \in \vec{W}_{II}$ find $W_3(W_2)$ such that

$$\mathcal{P}(H^*(W_2), W_2, W_3) = C.$$

Note that $W_{III} = W_3(W_{II})$. Call this collection of points $\vec{W}_{III}$.

2. For each $W_3 \in \vec{W}_{III}$ construct function $V^*(W)$ as follows:

$$V^*(W_3) = C + \mu\theta(H^*(W_2) - 1) + \mu\beta(V^*(W_2) - C).$$

where W_2 is such that $W_3 = W_3(W_2)$.

3. For each $W_3 \in \vec{W}_{III}$ construct function $H^*(W)$ as the solution to the following indifference condition:

$$V^*(W_3) = \max_{W_n > W_2} \left\{ \begin{array}{c} \mathcal{P}(H, W_n, W_3)) + \mu\overline{\theta}(H-1) + \mu\beta(\tilde{V}(W_n) - C) \\ s.t.\ \mathcal{P}(H, W_n, W_3) \leq C \end{array} \right\}$$

where W_2 is such that $W_3 = W_3(W_2)$ and $\tilde{V}(W_n)$ is a shape preserving (cubic Hermite) interpolant of $V^*(W)$ on $[W_{II}, \mathcal{W}(H^s)]$.

Note here that we do not require that the wealth levels in $\vec{W}_{III}$ are less than W_{II}. Even if an element of $\vec{W}_{III}$ is larger than W_{II}, we keep it, as it provides an additional point where we constructed the values of V^* and H^*. This implies there is an asymmetry between Leg 2 and Leg 3 because in Leg 2 we required all wealth levels in $\vec{W}_{II}$ to be less than W_I. The reason for this asymmetry is to ensure that the initial range $[W_I, \mathcal{W}(H^s)]$ where we use the Taylor approximation to construct $V^*$and hence, H^*, is small relative to the "length of the step" between the values of W in successive Legs.

4.

- Verify that $H^*(W)$ is increasing, that $H^*(W) > \mathcal{H}(W)$, and that $V(W)$ is increasing and concave on $[W_{III}, \mathcal{W}(H^s)]$. To do this, check that for any point A on $\vec{W}_{III} \cup \vec{W}_{II} \cup \vec{W}_I$

  $$(1+\epsilon)H^*(A) > \mathcal{H}(A);$$

  check that for any two adjacent points on $\vec{W}_{III} \cup \vec{W}_{II} \cup \vec{W}_I$, $A < B$,

  $$(1+\epsilon)H^*(B) \geq H^*(A) \text{ and } (1+\epsilon)V^*(B) \geq V^*(A);$$

  and check that for any three adjacent points on $\vec{W}_{III} \cup \vec{W}_{II} \cup \vec{W}_I$, $A < B < C$,

  $$(1+\epsilon)V^*(B) \geq \alpha V^*(A) + (1-\alpha)V^*(C),$$

  where $\alpha = \frac{B-A}{C-A}$ and $\epsilon = 10^{-4}$.

  Note that by construction we have that $H^*(W(H^s)) = H^s$ and that the indifference condition (34) is satisfied for all W's on the grid.

- If not, find $\underline{W} \in [W_{III}, W_{II}]$ such that the four conditions above are satisfied on $[\underline{W}, W^s]$ and stop. Equilibrium exists for all $H_0 \geq \max(H^*(\underline{W}), \underline{H})$.

Leg 4. etc... Repeat the previous leg as long as $H^*(W_{III}) > \underline{H}$.

13.1.2 Facts for the Numerical Procedure

Fact A.4.1. *Suppose that Assumptions 1-2 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Suppose that the function $V^*(W)$ defined in (32) has a kink at $\mathcal{W}(H^s)$. Then, if $H_0 < H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}$, it must be the case that $W_n(H_0) < \mathcal{W}(H^s)$. Moreover, for all $H \in [H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}, H^s]$, $W_n(H) = \mathcal{W}(H^s)$.*

Proof of Fact A.4.1. From the proof of the Theorem (see, in particular, the discussion following Fact A.2.7) we know that $W_n(H)$ must either satisfy the first order condition

$$\mu \frac{dV^*(W_n(H))}{dW} = \frac{1}{H} \tag{89}$$

or equal $\mathcal{W}(H^s)$ if it is the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} \geq \frac{1}{H}.$$

To prove the first statement, we show that if $H_0 < H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}$

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} < \frac{1}{H_0}. \tag{90}$$

To establish this, we make use of three preliminary results.

Claim A.4.1. *If $V^*(W)$ has a kink at $\mathcal{W}(H^s)$, it must be the case that $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s]$, $\underline{H} \in [H_0, H^s]$.*

Proof of Claim A.4.1. If $V^*(W)$ kinks at $\mathcal{W}(H^s)$, then we have that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H^s} > \lim_{W \searrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H^s} = 0.$$

By continuity, for H's smaller than but close to H^s we have

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H} > 0 > \lim_{W \searrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H},$$

implying that $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s]$. ■

Claim A.4.2. *Assuming that $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s]$, the left derivative of the function $\omega(W) \equiv W^*(H_c(W))$ at $\mathcal{W}(H^s)$ is*

$$\lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) = \frac{1}{\beta(1+\mu)}.$$

Proof of Claim A.4.2. Note from Fact A.4.2 below that, since $W_n(H) = \mathcal{W}(H^s)$ on the interval $[\underline{H}, H^s]$, it follows that

$$F(\omega(\omega(W)), \omega(W), W) = 0, \tag{91}$$

where the function F is defined below. Differentiating (91) with respect to W, we obtain

$$F_1(\omega(\omega(W)),\omega(W),W)\omega'(\omega(W))\omega'(W)+F_2(\omega(\omega(W)),\omega(W),W)\omega'(W)$$

$$+F_3(\omega(\omega(W)),\omega(W),W)=0.$$

Using the fact that $\omega(\mathcal{W}(H^s))=\mathcal{W}(H^s)$, we have that

$$F_1\left[\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)\right]^2+F_2\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)+F_3=0 \tag{92}$$

where F_i is the partial derivative evaluated at $(\mathcal{W}(H^s),\mathcal{W}(H^s),\mathcal{W}(H^s))$.

We know from the discussion prior to Fact A.4.2 that

$$F_1=\left(\frac{\mu\overline{\theta}}{1-\mu\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\right)\left(\frac{\beta-\beta\mu^2}{1-\beta}\right),$$

$$F_2=\left(\frac{\mu\overline{\theta}}{1-\mu\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\right)\left(\frac{\mu-\beta+\mu^2\beta-1}{1-\beta}\right),$$

and

$$F_3=\left(\frac{\mu\overline{\theta}}{1-\mu\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\right)\left(\frac{1-\mu}{1-\beta}\right).$$

Thus, (92) simplifies to

$$\left(\beta-\beta\mu^2\right)\left[\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)\right]^2+\left(\mu-\beta+\mu^2\beta-1\right)\left[\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)\right]+1-\mu=0$$

The relevant root is

$$\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)=\frac{1-\mu}{\beta-\beta\mu^2}=\frac{1}{\beta(1+\mu)}.$$

■

Claim A.4.3. *Assuming that $W_n(H)=\mathcal{W}(H^s)$ on some interval $[\underline{H},H^s]$, the left derivative of the function $V^*(W)$ at $\mathcal{W}(H^s)$ is*

$$\lim_{W\nearrow\mathcal{W}(H^s)}\mu\frac{dV^*(W)}{dW}=\mu^2\overline{\theta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\left(\frac{1}{1-\beta}\right).$$

Proof of Claim A.4.3. For $W\in[W^*(\underline{H}),\mathcal{W}(H^s)]$, we have that

$$V^*(W)-C-\frac{u}{1-\beta}=\mu\overline{\theta}(\mathcal{H}(\frac{W-\beta\omega(W)}{1-\beta})-1)+\mu\beta\left(V^*(\omega(W))-C-\frac{u}{1-\beta}\right).$$

Differentiating, taking the limit as $W\nearrow\mathcal{W}(H^s)$, and using the fact that $\omega(\mathcal{W}(H^s))=\mathcal{W}(H^s)$, we have that

$$\lim_{W\nearrow\mathcal{W}(H^s)}\frac{dV^*(W)}{dW}=\mu\overline{\theta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}(\frac{1-\beta\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)}{1-\beta})+\lim_{W\nearrow\mathcal{W}(H^s)}\mu\beta\frac{dV^*(W)}{dW}\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W).$$

It follows from this equation that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW} = \mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\left(\frac{1-\beta\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)}{1-\mu\beta\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)}\right). \tag{93}$$

It follows from Claim A.4.2 that

$$\begin{aligned}\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW} &= \mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\left(\frac{1-\frac{\beta}{\beta(1+\mu)}}{1-\frac{\mu\beta}{\beta(1+\mu)}}\right)\\ &= \mu\overline{\theta}\frac{\mu}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}.\end{aligned}$$

■

Given Claim A.4.3, to establish (90), we therefore need to show that

$$\mu^3\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} < \frac{1}{H_0}.$$

We know from the definition of H^s that

$$\frac{1}{H^s} = \left(\frac{\mu}{1-\mu\beta}\right)\mu\overline{\theta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}$$

This implies that

$$\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} = \frac{1-\mu\beta}{\mu^2\overline{\theta}H^s}$$

and hence that[40]

$$\mu^3\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} = \frac{\mu\,(1-\mu\beta)}{H^s(1-\beta)}.$$

Thus, since

$$\frac{1}{H_0} > \frac{\mu\,(1-\mu\beta)}{H^s(1-\beta)},$$

$$\mu^3\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} < \frac{1}{H_0}, \tag{94}$$

as required.

For the second statement, note that following the same argument used to establish (90), if $H > H^s\frac{(1-\beta)}{\mu(1-\mu\beta)}$, then it must be the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu\frac{dV^*(W)}{dW} > \frac{1}{H}. \tag{95}$$

[40] This also implies that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW} = \frac{1-\mu\beta}{1-\beta}\frac{1}{H^s}$$

This implies that for such H, $W_n(H) = \mathcal{W}(H^s)$. ■

Our next result concerns the function $F(W_1, W_2, W_3)$ which is defined as follows:

$$F(W_1,W_2,W_3) \equiv \frac{\beta W_2 - W_3 - (\beta\mathcal{W}(H^s) - W_2)}{\mathcal{H}(\frac{W_3-\beta W_t}{1-\beta})} + \mu\overline{\theta}(\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta}) - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{u}{1-\beta}\right)$$
$$-\left[\mu\overline{\theta}(\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta}) - 1) + \mu\beta\left(\begin{array}{c}\frac{\beta W_1 - W_2 - (\beta\mathcal{W}(H^s) - W_1)}{\mathcal{H}(\frac{W_t-\beta W_1}{1-\beta})} \\ +\mu\overline{\theta}(\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta}) - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{u}{1-\beta}\right)\end{array}\right)\right].$$

This function is well defined and differentiable on the domain $[\mathcal{W}(H^s) - \epsilon, \mathcal{W}(H^s)]^3$ for ϵ sufficiently small. Moreover, the functions' partial derivatives are

$$F_1(W_1,W_2,W_3) = \mu\overline{\theta}\frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW} + \mu^2\overline{\theta}\frac{\beta^2}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}$$
$$-\mu\beta\left(\frac{(\beta+1)\,\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta}) + \frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}\frac{\beta}{1-\beta}\left(\beta W_1 - W_2 - (\beta\mathcal{W}(H^s) - W_1)\right)}{\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})^2}\right),$$

$$F_2(W_1,W_2,W_3) = \frac{(\beta+1)\,\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta}) + \frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW}\left(\beta W_2 - W_3 - (\beta\mathcal{W}(H^s) - W_2)\right)}{\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})^2}$$
$$-\mu\overline{\theta}\frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW} - \mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW} - \mu^2\overline{\theta}\frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}$$
$$+\left[\mu\beta\left(\frac{\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta}) + \frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}\frac{1}{1-\beta}\left(\beta W_1 - W_2 - (\beta\mathcal{W}(H^s) - W_1)\right)}{\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})^2}\right)\right],$$

and

$$F_3(W_1,W_2,W_3) = -\left(\frac{\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta}) + \frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW}\frac{1}{1-\beta}\left(\beta W_2 - W_3 - (\beta\mathcal{W}(H^s) - W_2)\right)}{\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})^2}\right)$$
$$+\mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW}.$$

Letting $F_i = F_i(\mathcal{W}(H^s), \mathcal{W}(H^s), \mathcal{W}(H^s))$ and using the fact that

$$\mu^2\overline{\theta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\left(\frac{1}{1-\mu\beta}\right) = \frac{1}{H^s}, \tag{96}$$

these expressions imply that

$$F_1 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\right)\left(\frac{\beta - \beta\mu^2}{1-\beta}\right),$$

$$F_2 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\right)\left(\frac{\mu - \beta + \mu^2\beta - 1}{1-\beta}\right),$$

and that

$$F_3 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\right)\left(\frac{1-\mu}{1-\beta}\right).$$

Fact A.4.2. *Suppose that Assumptions 1-2 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Suppose that the function $V^*(W)$ defined in (32) has a kink at $\mathcal{W}(H^s)$ and that on the interval $[\underline{H}, H^s]$ we have that $W_n(H) = \mathcal{W}(H^s)$. Define the sequence $\langle W_t(W)\rangle_{t=0}^{\infty}$ inductively by $W_0 = W$ and $W_t = W^*(H_c(W_{t-1}))$. Then, if $W \in [W^*(\underline{H}), \mathcal{W}(H^s)]$, it is the case that for all $t > 1$*

$$F(W_{t+1}(W), W_t(W), W_{t-1}(W)) = 0.$$

Proof of Fact A.4.2. For all t, the indifference condition implies that

$$V^*(W_t(W)) = \frac{1-\mu\beta}{1-\beta}\underline{u} + \mathcal{P}(H_t, \mathcal{W}(H^s), W_t(W)) + \mu\overline{\theta}(H_t - 1) + \mu\beta(V^*(\mathcal{W}(H^s)) - C).$$

where $H_t = H_c(W_{t-1})$. Moreover, from the definition of $V^*(W_t(W))$ in (32), we have that

$$V^*(W_t(W)) - C = \frac{1-\mu\beta}{1-\beta}\underline{u} + \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(V^*(W_{t+1}(W)) - C\right).$$

It follows that for all t, it must be the case that

$$\begin{aligned}
& \mathcal{P}(H_t, \mathcal{W}(H^s), W_t(W)) + \mu\overline{\theta}(H_t - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C\right) \\
= \; & \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(\frac{1-\mu\beta}{1-\beta}\underline{u} + \mathcal{P}(H_{t+1}, \mathcal{W}(H^s), W_{t+1}(W)) + \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta(V^*(\mathcal{W}(H^s)) - C)\right).
\end{aligned}$$

Moreover, we know that both $\mathcal{P}(H_t, W_t(W), W_{t-1}(W))$ and $\mathcal{P}(H_{t+1}, W_{t+1}(W), W_t(W))$ are equal to C and so we can write this equality as

$$\begin{aligned}
& \mathcal{P}(H_t, \mathcal{W}(H^s), W_t(W)) - \mathcal{P}(H_t, W_t(W), W_{t-1}(W)) + \mu\overline{\theta}(H_t - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta}\right) \\
= \; & \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(\begin{array}{c} \mathcal{P}(H_{t+1}, \mathcal{W}(H^s), W_{t+1}(W)) - \mathcal{P}(H_{t+1}, W_{t+1}(W), W_t(W)) \\ +\mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta}\right) \end{array}\right).
\end{aligned}$$

Using the pricing equation (29), we can rewrite this equality as

$$\frac{\beta W_t(W) - W_{t-1}(W) - (\beta \mathcal{W}(H^s) - W_t(W))}{\mathcal{H}(\frac{W_{t-1}(W) - \beta W_t(W)}{1-\beta})} + \mu\overline{\theta}(\mathcal{H}(\frac{W_{t-1}(W) - \beta W_t(W)}{1-\beta}) - 1)$$
$$+\mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta}\right)$$
$$= \mu\overline{\theta}(\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1)$$
$$+\mu\beta\left(\begin{array}{c} \frac{\beta W_{t+1}(W) - W_t(W) - (\beta \mathcal{W}(H^s) - W_{t+1}(W))}{\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})} + \mu\overline{\theta}(\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1) \\ +\mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta}\right) \end{array}\right).$$

This implies that

$$F(W_{t+1}(W), W_t(W), W_{t-1}(W)) = 0,$$

as required. ■

Fact A.4.3. *Suppose that Assumptions 1-2 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Then, if $H \in [H_0, H^s)$, it is the case that*

$$W^*(H_c(W^*(H))) < W_n(H).$$

Proof of Fact A.4.3. We begin by showing that

$$\mathcal{P}(H, W^*(H_c(W^*(H))), W^*(H)) > C. \tag{97}$$

Note that since $\mathcal{P}(H_c(W), W^*(H_c(W)), W) = C$, $H < H_c(W^*(H))$, and $\mathcal{P}$ is a continuous function, it is sufficient to show that

$$\frac{\partial \mathcal{P}(H, W', W)}{\partial H} < 0$$

at any solution of the equation $\mathcal{P}(H, W', W) = C$. Indeed, if the above inequality holds, for any W' and W, there cannot be more than one level of the housing stock at which $\mathcal{P}(H, W', W) = C$. To prove the inequality note from (29), that

$$\frac{\partial \mathcal{P}(H, W', W)}{\partial H} = -\overline{\theta} + S'(H) - \frac{(W - \beta W')}{H^2}.$$

Moreover, at a solution

$$-\frac{(W - \beta W')}{H} = \left[(1-H)\overline{\theta} + S(H) - C(1-\beta) - \underline{u}\right].$$

Thus, at a solution

$$\begin{aligned}
\frac{\partial \mathcal{P}(H,W',W)}{\partial H} &= -\overline{\theta}+S'(H)+\frac{\left[(1-H)\overline{\theta}+S\left(H\right)-C(1-\beta)-\underline{u}\right]}{H} \\
&= -\left[\frac{C(1-\beta)+\underline{u}-(1-2H)\overline{\theta}-HS'(H)-S\left(H\right)}{H}\right] \\
&= -\left[\frac{\underline{u}+H\overline{\theta}-\left((1-H)\overline{\theta}+S(H)+HS'(H)-C(1-\beta)\right)}{H}\right]<0.
\end{aligned}$$

where the last inequality follows from Assumptions 1(i) and 2.

By definition, we know that

$$\mathcal{P}(H,W_n(H),W^*(H))\leq C.$$

Thus, given (97), $W_n(H)$ must be larger than $W^*(H_c(W^*(H)))$ as required. ■