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Community Development with Externalities and Corrective Taxation

Abstract

The decision to build in a community is associated with both positive and negative externalities. The textbook remedy is a tax or subsidy on new construction. This paper studies the political determination of such a policy in a model of community development with externalities. The commitment solution in which the community's initial residents choose policy for all periods is time inconsistent. This produces interesting development paths when policies are chosen each period by the current residents. On the one hand, an equilibrium exists in which policies move gradually in a pro-development direction, resulting in falling housing prices and increasing community size. On the other, there exist equilibria in which development is stalled, producing permanently high housing prices. This means that, in the long run, policy can be close to optimal or much too restrictive. Accordingly, for a broad range of initial conditions, corrective taxation can increase or lower social welfare.

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1 Introduction

Community development is an important topic in public and urban economics. The large literature on the subject identifies a number of potential externalities associated with the decision to build a new house in a community. On the positive side, there are externalities associated with the cost-sharing of public goods and agglomeration economies more generally. On the negative side, there are externalities arising from congestion or free-riding on existing public assets. These externalities mean that leaving development to the free market is unlikely to produce optimally-sized communities.

In principle, these externalities can be tackled with a variety of different policy instruments. The textbook remedy is a Pigouvian tax or subsidy on new construction. Appropriately set, such a tax or subsidy can align private and social incentives and assure that communities develop optimally. In the presence of negative externalities, growth-control measures such as land-use and building regulations, zoning, and limiting building permits can also work.[1] Such policies create a "regulatory tax" (Glaeser, Gyourko, and Saks 2005b) which acts like a Pigouvian tax to dampen development. With positive externalities, communities can offer tax abatements to developers or make infrastructure improvements which implicitly subsidize new development.

Public choice theory teaches us that we need to understand how a policy instrument will be determined by the political process, before we can judge whether it will actually improve social welfare. This lesson has spawned a literature studying the political determination of these development-regulating policies in various models of community development. Interest in this literature comes not only from the normative question of the social benefit of such policies, but also the desire to better understand the positive features of community development. A body of empirical work suggests that local regulations that control growth are important influences on the dynamics of housing supply and prices.[2]

Early work in this literature employs static, multi-community models in which communities first select policies and then households choose where to live. Land is typically assumed owned by absentee landlords and households are renters. Policies are chosen with a close eye on landlords' rents. While yielding useful insights, these models do not capture the incentives of current

[1] Rosen and Katz (1981) provide a well-known description of the various growth-control measures used by communities in the San Francisco Bay Area.

[2] A good review of this evidence is provided by Gyourko and Molloy (2015).

homeowners to promote the value of their properties. Such incentives seem important in practice. Indeed, in his influential book the *Homevoter Hypothesis*, William Fischel argues that protecting home values is the key motivator of local government behavior (Fischel 2001). More recent work has therefore turned to dynamic models which try to capture the redistributional conflict between current and future homeowners. To keep things tractable, these models typically focus on the decisions of a single community.

This paper contributes to this literature by studying the political determination of a Pigouvian tax or subsidy in a dynamic model of a single community in which homeowning residents choose policy and externalities can be positive or negative. The analysis produces a number of novel insights. First, it identifies a time inconsistency problem in that future residents would like to develop the community beyond the level that would result if the initial residents could choose policies for all future periods. Second, it shows that this time inconsistency problem results in interesting dynamic patterns of development when policies are chosen in each period by the current residents. On the one hand, an equilibrium exists in which policies are gradually moved in a pro-development direction, resulting in falling housing prices and increasing community size. On the other, there exists equilibria in which development is stalled, producing permanently high housing prices. Third, the range of equilibria that can arise with sequential policy-making means that, in the long run, the tax or subsidy can be set close to its optimal level or be set so high that the community ends up considerably undersized. This implies that, for a broad range of initial conditions, welfare when residents have access to corrective taxation can be higher or lower than with free entry.

The model community studied in this paper starts out with a stock of housing and initial residents who own this housing. The community can grow by building new housing and new construction is supplied by competitive developers. There is a pool of potential residents with heterogeneous desires to live in the community, generating a downward-sloping demand curve. There is turnover, with households entering and exiting the pool each period, so that the market for housing is always active. The possibility that current residents may leave the pool makes them care about the value of their homes. When living in the community, all residents obtain a common payoff or "surplus" which depends upon the number of residents. This creates an externality which can be either positive or negative depending on whether surplus is increasing or decreasing in the number of residents. The community can levy a tax or subsidy on new construction and the revenues or costs are shared equally by all residents. Policy decisions are made by the residents who are fully forward-looking.

The development plan that would emerge if the initial residents could choose policies for all periods takes one of two possible forms. If the initial housing stock exceeds a threshold level, there is no development. To achieve this outcome, the residents impose a tax on new construction sufficient to choke off demand. If the initial housing stock is below the threshold, new construction is provided in the initial period and thereafter there is no further development. The extent of development is smaller than socially optimal. If the externality is negative, new construction is taxed but the tax is set too high. If the externality is positive, new construction is subsidized only under restrictive conditions. These are that the externality be sufficiently large, the initial housing stock be not too small, and the probability that residents leave the community be sufficiently low. When these conditions are not satisfied, new construction is taxed. Even when it is subsidized, the subsidy is set too low.

This *commitment solution* is time inconsistent when it involves development. Once the population has expanded in the initial period, next period's residents would want to expand it further. The reason they would favor such an expansion while the initial residents would not, is that they do not internalize the negative consequences for the initial period.

When policy is chosen each period by the current residents there exists an *equilibrium with gradual development* in which the community's housing stock converges asymptotically to a steady state. This steady state is higher than the maximum housing level that can arise in the commitment solution but is still smaller than socially optimal. However, it approaches the social optimum as the probability residents leave the community vanishes. In this equilibrium, if the externality is negative, new construction is taxed, but the tax is declining through time which is what keeps the community growing. If the externality is positive, residents will offer subsidies on new construction under relatively weak conditions. However, when the initial housing stock is small, subsidies are only offered once the community has grown sufficiently large. When they are offered, subsidies will be increasing through time.

There also exist *equilibria with stalled development* in which the community's housing stock expands in the initial period and thereafter there is no further development. The reason is that residents understand that expanding beyond this level will precipitate a large growth of housing which will have negative implications for the housing price that offset any benefits from tax revenue or from a higher population. For a range of initial housing levels, there exist equilibria of this form in which the steady state housing level is *lower* than that arising in the commitment solution.

Whether giving residents the power to impose corrective taxation increases social welfare depends on the scale of the externality and which equilibrium arises. For small values of the externality, free entry dominates unambiguously. For larger values, welfare can be higher with corrective taxation if the equilibrium with gradual development arises or lower if an equilibrium with stalled development results. Moreover, when the externality is positive, even if the equilibrium with gradual development results, free entry can dominate when the initial housing stock is small. This is because the community spends a long time taxing new construction in the transition to the steady state.

The organization of the remainder of the paper is as follows. Section 2 discusses related literature. Section 3 outlines the model. Section 4 establishes two benchmarks by describing socially optimal development and development with free entry. Section 5 explains the commitment solution and discusses its time consistency. Section 6 explores development with sequential policy-making and describes the different types of equilibria that arise. Section 7 addresses the question of how enabling residents' to impose corrective taxation impacts social welfare. Section 8 concludes by summarizing the main lessons learned.

2 Related literature

This paper contributes to the literature studying the political determination of policies that seek to regulate community development. As noted in the introduction, early papers in this literature employed static, multi-community models. A nice example is Helsley and Strange (1995) who consider a spatial model with competing communities.[3] A fixed population of households with identical preferences choose in which community to live and rent a house. Houses are of uniform size and there is a "passive" community available which does not regulate its growth and serves as a repository for households crowded out of the other communities. In each community, higher populations create negative externalities via congestion. The paper considers both the case in which communities regulate their size by choosing their boundaries and that in which they employ entry fees (taxes). Communities are owned by profit-maximizing absentee landowners and choose

[3] Other good examples are Brueckner (1995) and Brueckner and Lai (1996).

their policies to maximize landowner profits. Entry fee revenues are included in these profits. Helsley and Strange show that, with either policy, communities choose to deter entry and this reduces aggregate welfare.

A more recent paper in this vein is Hilber and Robert-Nicoud (2013) who develop a multi-community model in which each community's planning board chooses a regulatory tax. In contrast to prior work, households have heterogeneous preferences over communities. Moreover, some communities have better amenities than others and higher populations create negative externalities via congestion. Planning boards choose their regulatory taxes to maximize land rents plus tax revenues. In equilibrium, communities with better amenities have higher populations and impose higher regulatory taxes. Empirical support for this prediction is provided using cross-sectional data from U.S. metro areas.

Fernandez and Rogerson (1997) and Calabrese, Epple, and Romano (2007) study the determination of zoning in static multi-community models in the Tiebout (1956) tradition. Communities provide public services which are financed by property taxation. Households differ in their incomes and public services are normal goods. Households rent housing from absentee landlords and zoning is modelled as a minimum level of housing consumption. Residents choose zoning along with the level of public services for their communities and are bound by the rules they impose. The motivation for zoning is to prevent low income households from joining the community, renting small houses, and free riding on the property tax financed public services. Zoning arises in equilibrium and facilitates income stratification as envisaged by Hamilton (1975). Welfare implications are complex, as zoning redistributes from low to high income households, but lessens distortions in housing consumption and public service provision.

Turning to the work analyzing dynamic models of development regulation, Glaeser, Gyourko, and Saks (2005a) analyze the one-time, irreversible decision of a zoning authority to approve a development project seeking to build a fixed number of new homes in a town. The population that would live in these new homes would reduce the welfare of the existing residents because of a congestion externality. Existing residents live in the town for a finite number of periods, after which they sell their homes to a new set of potential residents.[4] The project reduces the price of these homes at time of sale by increasing supply, which provides another reason for existing

[4] The model also features an exogenous fraction of renters, but these play no role in influencing the zoning authority's decision.

residents to oppose the project. The developer obtains the rents created by selling the new homes and thus wants to see it approved. The zoning authority has its own preferences over whether the project should be approved, but is also swayed by the time and money the existing residents and developer devote to influencing it. The paper solves for the influence activities of the existing residents and the developer and computes the equilibrium probability that the project is approved. This is then related to the exogenous parameters of the model (for example, the zoning authority's preference and the costs of influence activities) and the implications are discussed in light of U.S. experience.

Ortalo-Magne and Prat (2014) study the supply of building permits in an ingenious dynamic general equilibrium model with a single city and a rural area. The model has overlapping generations of citizens who choose, when young, whether to locate in the city or the countryside. If they locate in the city, they have to decide whether to rent or own a home. Housing is infinitely durable and building new homes in the city requires developers to have building permits. In each period, the residents of the city choose how many permits to issue. Residents receive an exogenous fraction of the rents the permits create for developers. The paper characterizes the smallest city size which supports an equilibrium in which no additional permits are issued and shows that this size is smaller than optimal. There is no externality in the model, so there is no welfare rationale for permits. The result that housing can be under-supplied is driven by the incentives of home-owners to increase the value of their homes by restricting supply. Home-owners care about the value of their homes because they sell them in the last period of their lives.

Barseghyan and Coate (2016) study zoning in a dynamic two-community model in the Tiebout tradition with two types of houses - small and large. Potential residents choose in which community to live and what size of house to buy. Housing is imperfectly durable, so housing stocks in a community can shrink. In each period, residents choose the level of a public service which is financed by a property tax and whether or not to impose a zoning regulation that requires all newly constructed houses to be large. The paper finds an equilibrium in which both communities always impose zoning, so that the housing stock converges to all large houses and citizens end up overconsuming housing. The desire of residents to impose zoning reflects the incentive of the owners of small homes to raise the value of their homes by restricting supply and the incentive of large home-owners to prevent others from building small homes in their communities and free-riding on public service provision.

All the work reviewed here shares a common focus and the present paper follows squarely in this tradition. In some respects, the model analyzed here is simpler than in much of the literature. There is a single community, a single policy instrument, and the policy is determined solely by the interests of the residents - all of whom own identical homes.[5] On the other hand, the policy is continuous and can be adjusted as the community develops. This allows focus on the evolution of the policy over time and its impact on the dynamics of development. In particular, gradual development arises from the fact that policies are moved in a pro-development direction through time. The model is also distinctive in allowing for both positive externalities and subsidies. Positive externalities feature prominently in the urban economics literature on agglomeration economies and are a motivation for so-called "place-making policies".[6] The literature on such policies focuses on the desirability of intervention by higher levels of government to subsidize development projects in localities, but it seems important to first understand the incentives of residents to subsidize their own community's development. Finally, this paper offers a more nuanced answer to the welfare question than in much of the literature: specifically, in many circumstances, development-regulating policies can lead to a higher or lower level of welfare than free entry.

The present paper is a close cousin to Barseghyan and Coate (2018) who introduce the basic model of community development used here. Their community provides a durable local public good which can be financed with debt and taxation. In each period, the residents choose how much to invest in the public good and how to finance this investment. The paper finds an equilibrium in which, for some initial conditions, the community gradually develops via public wealth accumulation. The community's public wealth is the difference between the value of its public good stock and its debt. A higher level of public wealth allows the community to offer a higher public good surplus. Increasing public wealth via tax-financed increases in the public good or debt reductions, thus attracts more residents. Residents' incentive to build wealth depends

[5] In particular, in the model of this paper, the interests of developers are not relevant because they earn zero profits from new construction. This reflects the assumption that the residents get the revenues if a tax is imposed. With a quantity control such as no-fee building permits, developers obtaining permits would earn profits and it would be natural to assume they would lobby to get permission to build (as in Glaeser, Gyourko, and Saks 2005a). We do not look at the no-fee permit versus tax comparison here because we are also interested in considering positive externalities and subsidies, but we conjecture that, in the negative externality case, similar results would arise if permits were issued by a planning board who weighed residents' welfare and developer profits.

[6] For good discussions of place-making policies see Glaeser and Gottlieb (2008) and Neumark and Simpson (2015).

on the community's initial wealth and the strength of the cost-sharing externality created by the public good. Development via public wealth accumulation is inefficient in that it both involves delay and leads the community to be undersized, but it does allow the community to develop even with unfavorable initial conditions.

The key difference between this paper and Barseghyan and Coate (2018) is that residents can tax or subsidize new construction. Moreover, to focus cleanly on corrective taxation, this paper abstracts from public goods, debt, and general taxation, and models externalities in a reduced form manner. Some of the results have parallels in Barseghyan and Coate (2018) because public wealth accumulation has similarities to a subsidy. In particular, a time inconsistency problem also arises with public wealth accumulation and an equilibrium with gradual development exists. Nonetheless, the mechanisms are distinct, since a subsidy works differently than wealth accumulation.[7] In addition, this paper features an equilibrium with stalled development. Finally, the point of the papers are different: this paper analyzes how corrective taxation impacts development in the presence of externalities and whether it improves welfare. Barseghyan and Coate (2018) introduces the idea of community development by public wealth accumulation.

From a theoretical perspective, the problem studied in this paper relates to the classic durable good monopoly problem analyzed by Bulow (1982) and Stokey (1981). In that problem, a monopoly producer of a durable good decides how much to produce in each period. There is no commitment, so that the producer has an incentive to gradually expand production to reach consumers with lower valuations of the good. However, consumers are forward-looking and can time their purchases. The key questions concern the equilibrium time path of prices and output and how close total production gets to the socially efficient level. The residents' problem is related, but differs in at least three interesting ways. First, residents care not just about revenues from taxing new construction but also the value of their homes (i.e., the value of goods already produced). Second, the set of decision-makers is changing over time as the community expands. Third, because residents live in the community, they care about the externalities from new construction.

Finally, the paper contributes to the growing literature developing and analyzing infinite hori-

[7] Wealth accumulation requires residents to invest in the present to create incentives in the future. A subsidy works immediately and is paid for contemporaneously. This raises the question of how the two mechanisms compare. We leave this for future work because, while feasible, incorporating both development strategies in the same model makes for a much more involved analysis.

zon political economy models of policy-making with forward-looking decision makers.[8] Many interesting issues arise from recognizing the dynamic linkage of policies across periods. Such linkages arise directly, as with public investment or debt, or indirectly because current policy choices impact citizens' private investment decisions. The model studied here features a state variable determined by the market (the housing stock) but shaped by residents' policy decisions. It also features a changing group of decision-makers, as the size of the community is growing.

3 Model

Consider a community such as a small town or village. This community can be thought of as one of a number in a particular geographic area. The time horizon is infinite and periods are indexed by $t = 0, ..., \infty$. There is a pool of potential residents of size 1. These can be thought of as households who for exogenous reasons (employment opportunities, family ties, etc.) need to live in the geographic area in which the community is located and are potentially open to living in the community. Potential residents are characterized by their desire to live in the community (as opposed to an alternative community in the area) which is measured by the preference parameter θ. This desire, for example, may be determined by a household's idiosyncratic taste for the community's natural amenities. The preference parameter takes on values between 0 and $\overline{\theta}$, and the fraction of potential residents with preference below $\theta \in [0, \overline{\theta}]$ is $\theta/\overline{\theta}$. Reflecting the fact that households' circumstances change over time, in each period new households join the pool of potential residents and old ones leave. The probability that a household currently a potential resident remains one in the subsequent period is $\mu \in (0, 1]$. Thus, in each period, a fraction $1 - \mu$ of households leave the pool and are replaced by an equal number of new ones.

The only way to live in the community is to own a house. The community has sufficient land to accommodate housing for all the potential residents. Moreover, the only use for land is building houses.[9] Houses are infinitely durable and the cost of building a new one is C.[10] Housing is supplied by competitive developers. The stock of houses at the beginning of a period is denoted

[8] Examples of this style of work are Azzimonti (2011), Battaglini and Coate (2008), Bowen, Chen, and Eraslan (2015), Brinkman, Coen-Pirani, and Sieg (2018), Coate and Morris (1999), Hassler, Rodriguez Mora, Storesletten, and Zilibotti (2003), and Krusell and Rios-Rull (1999).

[9] We could alternatively assume that land not used for housing has a constant productivity in agricultural use.

[10] The assumption of infinitely durable housing is common in the urban economics literature and is justified by the fact that buildings in developed countries display considerable longevity.

by H and the stock at the beginning of the next period by H'. New construction is therefore $H' - H$. A stock of housing H can accommodate a fraction H of the pool of potential residents. The initial housing stock is denoted H_0.[11]

A competitive housing market operates in each period. Demand comes from new households moving into the community, while supply comes from owners leaving the community and new construction. The community levies a tax τ on new construction. This tax is paid by developers on every new house they build. The tax τ can be negative, which would mean that the community was subsidizing new construction. The revenues/costs of the tax/subsidy are divided equally between all households who reside in the community at the end of the period. Thus, each household receives an amount $\tau(H' - H)/H'$. The price of houses is denoted P. The determination of the housing price and level of new construction will be discussed in more detail below.

When living in the community, a household with preference parameter θ and consumption x obtains a period payoff of $\theta + x + S(H')$ if the number of households is H'. The function $S(H')$ represents the surplus associated with living in the community. Intuitively, this surplus can be thought of as determined by the net benefits of the public spending undertaken by the community on behalf of its residents, along with population-related costs and benefits such as congestion or beneficial social interactions. We assume that over the relevant range of housing levels, $S(H)$ is equal to $S+sH$ where S is non-negative and s can be positive or negative. The sign of s determines whether there is a positive or negative externality associated with higher population. This linear specification allows us to both obtain analytical solutions and capture the direction and strength of the externality in a simple way. Some possible micro-foundations for such a linear net surplus function are offered in the on-line Appendix.

Households discount future payoffs at rate β and can borrow and save at rate $\rho = 1/\beta - 1$. This assumption means that households are indifferent to the intertemporal allocation of their consumption. Each household receives an exogenous income stream the present value of which is sufficient to purchase housing in the community and to pay any tax obligations.[12] When not living in the community, a household's per period payoff (net of the consumption benefits from

[11] It is important that H_0 be positive, so the community has residents. If this were not the case, there would be nobody to choose the period 0 policies.

[12] The assumption that utility is linear in consumption means that there are no income effects, so it is not necessary to be specific about the income distribution.

income) is $\underline{u}$.[13]

The timing of the model is as follows. Each period, the community starts with a stock of houses H. At the beginning of the period, the existing residents choose the tax on new construction τ. Then, households who were in the pool of potential residents in the previous period learn whether they will be remaining and new households join. Those in the pool decide whether to live in the community and existing residents no longer in the pool prepare to leave it. The housing market opens and the equilibrium price P is determined along with new construction or, equivalently, next period's housing stock H'. New residents buy houses and move into the community and old ones sell up and leave. Developers pay taxes $\tau(H' - H)$. Residents enjoy a surplus $S(H')$ and share the revenues from the tax $\tau(H' - H)/H'$. Next period begins with the state H'.

3.1 Housing market equilibrium

We now explain how the housing market determines price and new construction. At the beginning of any period, households fall into two groups: those who resided in the community in the previous period and those who did not, but could in the current period. Households in the first group own homes, while the second group do not. Households in the first group who leave the pool sell their houses and obtain a continuation payoff of

$$P + \frac{\underline{u}}{1 - \beta}. \tag{1}$$

The remaining households in the first group and all those in the second must decide whether to live in the community. This decision will depend on their preference parameter θ, current and future housing prices, expected surplus, and their share of tax revenues/subsidy costs. Since selling a house and moving is costless, there is no loss of generality in assuming that all households sell their property at the beginning of any period.[14] This makes each household's location decision independent of its property ownership state. It also means that the only future consequences of the current location choice is through the price of housing in the next period.

Let $P(H')$ denote the anticipated equilibrium price of housing when the state is H'. Then, in a period in which the initial state is H, the new construction tax is τ, the price of housing is

[13] Note that $\underline{u}$ is both the per period payoff of living in one of the other communities in the geographic area if a household is in the pool and the payoff from living outside the area when a household leaves the pool.

[14] It should be stressed that this is just a convenient way of understanding the household decision problem. The equilibrium we study is perfectly consistent with the assumption that the only households selling their homes are those who plan to leave the community.

P, and households anticipate H' households living in the community, a household of type θ will choose to reside in the community if and only if

$$\theta + S(H') + \tau(H' - H)/H' - P + \beta P(H') \geq \underline{u}. \tag{2}$$

The left hand side of this inequality represents the per-period payoff from locating in the community (net of the consumption benefits of income), assuming that the household buys a house at the beginning of the period and sells it the next. The right hand side represents the per-period payoff from living elsewhere. Given (2) and the fact that household preferences are uniformly distributed over $[0, \overline{\theta}]$, the equilibrium price of housing P in the current period must satisfy the market clearing condition

$$H' = 1 - \frac{\underline{u} - (S(H') + \tau(H' - H)/H' - P + \beta P(H'))}{\overline{\theta}}. \tag{3}$$

This implies that the equilibrium price is

$$P = (1 - H')\overline{\theta} + S(H') + \frac{\tau(H' - H)}{H'} + \beta P(H') - \underline{u}. \tag{4}$$

On the supply side, the assumption that houses are infinitely durable implies that

$$H' \geq H. \tag{5}$$

Moreover, because the supply of new construction is perfectly elastic at a price equal to the construction cost plus the tax $C + \tau$, it must also be the case that

$$P \leq C + \tau \ (= \text{if } H' > H). \tag{6}$$

Given that next period's housing price is described by the function $P(H')$, any policy τ and housing market pair (H', P) is consistent with housing market equilibrium if and only if (4), (5) and (6) are satisfied.

3.2 Policy choice

Next we turn to residents' choice of policy. As explained above, the timing of the model is first that the existing residents choose the new construction tax τ, and then the housing market determines new construction H' and the price of housing P. Obviously, when residents choose policy they will anticipate how they impact the housing market. Rather than deriving the relationship between the housing market equilibrium and the tax, and then analyzing the optimal policy, it is easier to think of residents as directly choosing the housing price and new construction along with the policy, but subject to the constraint that their choice be consistent with housing market equilibrium. Thus, given any initial state H, we will assume that residents choose (τ, H', P) but subject to the market equilibrium constraints (4), (5), and (6).

Note further that there is no loss of generality in assuming that τ is chosen so that $\tau = P - C$. If $H' > H$, then this must be true in equilibrium. If $H' = H$, then no tax revenue is being raised and τ can be lowered to $P - C$ with no implications for the housing market equilibrium. This observation permits removing τ from the set of choice variables. It also permits ignoring constraint (6) since it will automatically be satisfied whether or not $H' > H$.

While residents differ in their desires to live in the community θ, they will have identical preferences over policy and hence there is no collective choice problem to resolve. To understand this, note that when the initial housing stock is H, existing residents will have preferences in the interval $[(1-H)\overline{\theta}, \overline{\theta}]$.[15] The market will allocate housing to those in the pool of potential residents with the highest θ and the supply of housing can only expand. It follows that residents will all anticipate living in the community as long as they remain in the pool. In light of this, the residents' objective function can be written as

$$(1-\mu)\left[P + \frac{u}{1-\beta}\right] + \mu\left[S(H') + \frac{(P-C)(H'-H)}{H'} + \beta V(H')\right], \tag{7}$$

where $V(H')$ measures a resident's continuation payoff. This reflects the fact that, with probability $1-\mu$, a resident leaves the pool and sells its house, and, with probability μ, they remain in the pool and continue to reside in the community. Note that the concrete interpretation of $V(H')$ is that it is the continuation payoff of a household with preference parameter $\theta = 0$ at the beginning of a period in which: i) the initial state is H'; ii) the household owns a house in the community but does not know whether they will remain in the pool; and iii) the household is constrained to live in the community as long as they remain in the pool. Residents actual continuation payoffs will include a constant term reflecting their desires to live in the community θ. The residents' problem is to choose a policy pair (H', P) to maximize (7) subject to the market equilibrium constraints (4) and (5).

[15] In periods $t = 1, ..., \infty$ this follows from the fact that, in equilibrium, the households with the highest preference for living in the community purchase houses in the community in the previous period. We assume that this condition also characterizes the initial distribution of residents in period 0.

3.3 Equilibrium defined

Given all this, an *equilibrium* consists of a housing rule $H'(H)$, a price rule $P(H)$, and a value function $V(H)$ solving the problem

$$V(H) = \max_{(H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H') + \frac{(P-C)(H'-H)}{H'} + \beta V(H')\right] \\ s.t.\ P = (1-H')\overline{\theta} + S(H') + \frac{(P-C)(H'-H)}{H'} + \beta P(H') - \underline{u} \ \&\ H' \geq H \end{array} \right\}. \tag{8}$$

The state space on which these functions must be defined is the interval of feasible housing levels $[H_0, 1]$.

4 Two benchmarks for comparison

This section describes two benchmarks with which we will compare the equilibrium: the development path that would be optimal for a utilitarian social planner and that which would arise with free entry (i.e., when residents do not have access to corrective taxation). The section also lays out the assumptions we will impose on the model's parameters.

4.1 Optimal community development

A utilitarian planner wishes to maximize the discounted sum of the aggregate payoffs of the different pools of potential residents. The assumption that utility is linear in consumption, implies that the planner is indifferent between transfers of consumption both between households in the same pool and across different pools. Accordingly, there is no loss of generality in simply assuming that, in any period, the cost of new construction is financed by lump-sum taxation of the pool of potential residents.

The planner's problem can be posed recursively. Given an initial stock of housing H, the planner chooses new construction or, equivalently, next period's housing stock H'. The planner must respect the feasibility constraint created by durable housing (5). The households in the pool with the highest θ will be allocated to the H' houses. Given that θ is uniformly distributed on $[0, \overline{\theta}]$, this implies that households in the interval $[(1-H')\,\overline{\theta}, \overline{\theta}]$ will be assigned to live in the community. Accordingly, the planner's problem is

$$U(H) = \max_{H'} \left\{ \begin{array}{c} \int_{(1-H')\overline{\theta}}^{\overline{\theta}} \theta \frac{d\theta}{\overline{\theta}} + H'S(H') + \underline{u}\,(1-H') - C(H'-H) + \beta U(H') \\ s.t.\ H' \geq H \end{array} \right\}. \tag{9}$$

The first two terms in the objective function represent the benefits received by the households assigned to the community, while the third term represents the benefits to those not so assigned. The fourth term represents the costs of new construction. The final term is the planner's continuation value.

There is no social benefit from delaying development, so that, if the community's initial housing stock is not so large to make the feasibility constraint bind, the solution will be to raise housing to the socially optimal level in the initial period. Observe that if the feasibility constraint is not binding, $U'(H)$ is equal to C. Differentiating the objective function, we see that the socially optimal housing level satisfies the first order condition

$$(1-H')\,\overline{\theta} + S(H') + H'S'(H') - C(1-\beta) - \underline{u} = 0. \tag{10}$$

The left hand side represents the net social benefit from assigning an additional household to the community. The term $(1-H')\,\overline{\theta}$ is the preference of the marginal household for living in the community and $S(H')$ reflects the surplus accruing to the marginal household. The term $H'S'(H')$ reflects the impact of adding the household on the surpluses of the other residents, and reflects the externality created by the additional household. The term $C(1-\beta)$ is the per-period cost of an additional house and the term $\underline{u}$ is the per-period payoff a household receives when not residing in the community. At the optimal housing level, this net social benefit is zero.

Using the assumption that $S(H)$ is linear, we can solve (10) for the socially optimal housing level. Letting this be denoted H^o, we have

$$H^o = \frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{\overline{\theta} - 2s}. \tag{11}$$

We now make assumptions on the parameters and the community's initial housing level to ensure that the planner's problem is well-behaved and that the feasibility constraint does not bind.

Assumption 1 (i)

$$s < \overline{\theta}/2.$$

(ii)

$$H_0 < H^o < 1.$$

Part (i) of the assumption implies that the net social benefit is decreasing in H and is necessary for the second order condition associated with (10) to be satisfied. Part (ii) simply implies that some development is socially optimal, but the community should not contain all the potential residents. Then, we have:

Proposition 1 *Under Assumption 1, the optimal community development plan is to construct $H^o - H_0$ new houses in the initial period. Thereafter, no more housing should be constructed.*

4.2 Equilibrium with free entry

If residents do not have access to a new construction tax or subsidy, there is nothing for residents to do and development is just determined by the market. Assuming that demand is sufficiently strong to induce some development, the price of housing will be C and all new construction will take place in the initial period. The equilibrium level of housing satisfies the condition that

$$(1 - H')\,\overline{\theta} + S(H') - C(1-\beta) - \underline{u} = 0. \tag{12}$$

The left hand side is the net private benefit obtained by the marginal household living in the community under the assumption that the price of housing is C. It differs from the net social benefit on the right hand side of (10) in not including the externality term $H'S'(H')$. Households will purchase housing until this net private benefit is equal to zero.

Using the assumption that $S(H)$ is linear, we can solve (12) for the free entry equilibrium housing level. Letting this be denoted H^e, we have

$$H^e = \frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{\overline{\theta} - s}. \tag{13}$$

We will make a further assumption to ensure that there will be some development with free entry.[16]

Assumption 2

$$H_0 < H^e < 1.$$

[16] It is certainly possible to do the analysis without this Assumption, but we make it to reduce the number of cases to be considered.

This assumption implies that, even if there is a positive externality, the private benefits of living in the community are sufficient to generate some new construction. Then, we have:

Proposition 2 *Under Assumption 2, community development with free entry involves the construction of $H^e - H_0$ new houses in the initial period. Thereafter, no more housing is constructed.*

The only difference between the socially optimal development plan and that which emerges with free entry, lies in the extent of new construction that occurs in the initial period. If there is a positive externality, the free entry housing level is too small, while if there is a negative externality, it is too large.

5 The commitment solution

We now characterize the plan that would be optimal for the initial residents of the community. This is interesting in its own right, since what happens with commitment is the usual starting point in dynamic policy problems. It is also useful for developing intuition for the case with sequential policy-making. In particular, understanding the time consistency of this plan, lends insight into how development will be modified in the sequential case.

Suppose then that the initial residents could commit the community to following a complete development plan. Such a plan would be described by $\{H_{t+1}, P_t\}_{t=0}^{\infty}$. Here H_{t+1} denotes the level of housing at the beginning of period $t+1$ and P_t the price of housing in period t. The optimal plan solves the problem[17]

$$\max_{\{H_{t+1},P_t\}_{t=0}^{\infty}} \left\{ \begin{array}{c} \sum_{t=0}^{\infty} (\mu\beta)^t \left\{ (1-\mu)\left[P_t + \frac{u}{1-\beta}\right] + \mu\left[S(H_{t+1}) + \frac{(P_t - C)(H_{t+1}-H_t)}{H_{t+1}}\right]\right\} \\ \text{s.t. for all } t \geq 0 \\ P_t = (1-H_{t+1})\overline{\theta} + S(H_{t+1}) + \frac{(P_t - C)(H_{t+1}-H_t)}{H_{t+1}} + \beta P_{t+1} - \underline{u} \;\&\; H_{t+1} \geq H_t \end{array} \right\}. \tag{14}$$

The difference between this and the equilibrium problem (8) is that the initial residents get to directly choose the entire sequence of policies rather than having only an indirect influence through their choice of next period's housing stock.

[17] To this problem we need to add the transversality condition that $\lim_{t\to\infty} \beta^t P_t = 0$. This prevents the initial residents using an increasing sequence of taxes to create a housing price bubble in which potential residents buy a house in the community expecting its price will rise in the next period. This price rise is caused by a higher future tax.

To describe the solution, let

$$H^* = \frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{\overline{\theta}(1 + \frac{1-\mu}{1-\mu\beta}) - 2s}. \tag{15}$$

How the initial housing level H_0 compares with H^* determines whether or not development takes place. Under Assumption 1, $H^* \leq H^o$ with the equality holding only when $\mu = 1$. Second, for all housing levels $H \leq H^*$, let

$$\mathcal{H}(H) = \frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{2\left(\overline{\theta} - s\right)} + \frac{\mu(1-\beta)\overline{\theta}}{(1-\mu\beta)\, 2\left(\overline{\theta} - s\right)} H. \tag{16}$$

This function describes the housing level that actually arises when development takes place (which is $\mathcal{H}(H_0)$). The function is linear with a positive intercept and a slope less than 1. Furthermore, $\mathcal{H}(H^*) = H^*$, so that for housing levels less than H^*, $\mathcal{H}(H) > H$. Finally, for all H, let

$$\mathcal{B}(H) = (1-H)\overline{\theta} + S(H) - C(1-\beta) - \underline{u}. \tag{17}$$

Recall from the previous section that $\mathcal{B}(H)$ is the net private benefit that would be obtained by the marginal household from living in the community if residents did not have access to a new construction tax or subsidy. We can now state:

Proposition 3 *Suppose that Assumptions 1 and 2 are satisfied. Then the commitment solution has the following form. (i) If $H_0 \geq H^*$, no new houses are provided. The price of housing in all periods is $C + \mathcal{B}(H_0)/(1-\beta)$ and new construction is taxed. (ii) If $H_0 < H^*$, $\mathcal{H}(H_0) - H_0$ new houses are provided in the initial period, and, thereafter, no more are provided. The price of housing is $C + \mathcal{H}(H_0)\mathcal{B}(\mathcal{H}(H_0))/H_0(1-\beta)$ in the initial period and $C + \mathcal{B}(\mathcal{H}(H_0))/(1-\beta)$ thereafter. If $s \leq \overline{\theta}(1-\mu)/(1-\mu\beta)$, new construction is taxed. Otherwise, new construction is taxed if $H_0 < \frac{\overline{\theta}-2s}{\overline{\theta}} \frac{1-\mu\beta}{\mu(1-\beta)} H^o$ and subsidized if the reverse inequality holds.*

Thus, the commitment solution takes one of two possible forms. If the initial housing stock exceeds H^*, there is no development. To achieve this outcome, the residents impose a time invariant tax that deters new construction.[18] In this case, the initial residents do not wish the community to develop because to do so would reduce the price of housing. The cost of the price reduction is not offset by the revenues raised by a lower tax and, in the case of a positive externality, the higher surplus that results from expansion.

[18] Residents could achieve the same outcome with a zoning ordinance that forbids the development of any undeveloped land.

If the initial housing stock is below H^*, there is development, all of which occurs in the initial period. If the externality is negative, new construction is taxed. The tax is higher in the initial period than thereafter because new residents share in the tax revenues the new construction generates. The initial period tax is set so that the benefits of additional tax revenue balance the costs of development in terms of the reduction in the price of housing and the lower surplus caused by the negative externality. If the externality is positive, new construction is subsidized only if the externality is sufficiently large and the initial housing stock not too small. Otherwise, it is taxed. If a subsidy is employed, it is higher in the initial period to account for the fact that new residents share the burden of financing the subsidy. It is set so that the additional benefits of expansion created for the initial residents by the surplus from the externality balance the costs in terms of the reduction in the housing price and the fiscal burden of the subsidy.

In either of its two forms, the commitment solution involves too little development. This is immediate when there is no development since the initial housing stock is smaller than optimal by Assumption 1(ii). When there is development, the result follows from the fact that $\mathcal{H}(H_0)$ is smaller than H^* which is in turn smaller than the optimal level H^o. Another way to frame this conclusion is that when there is a negative externality, the tax is set too high, and, when there is a positive externality, even when there is a subsidy, it is set too low.

The proof of Proposition 3 is in the Appendix. While the initial residents' problem (14) may look complicated, the proof shows it can be reduced to a problem involving only a single choice variable: the amount of development to undertake in the initial period. There are three features of the solution that seem particularly interesting. First, the initial residents are willing to subsidize new development only under restrictive conditions. In particular, note that, given Assumption 1(i), the requirement that s exceed $\overline{\theta}(1-\mu)/(1-\mu\beta)$ cannot be satisfied if the probability of remaining in the community μ is less than or equal to the discount rate β (which seems a reasonable expectation).[19] Thus, for subsidization to occur, the residents must have a very high probability of remaining in the community, in addition to the externality being large and the community not being too small. The reason community size matters is that the per capita cost of the subsidy for the initial residents is higher when the community is smaller.

Second, when the community subsidizes development, all the subsidization is done in the initial period rather than being spread over time. One might have guessed that the initial residents

[19] This is because $(1-\mu)/(1-\mu\beta)$ is greater than or equal to $1/2$ when μ is less than or equal to β.

would delay some of the subsidization until they had attracted more residents, thereby spreading the burden over a larger population. However, this turns out not to be desirable because any subsidization that potential residents have to contribute to in the future will be anticipated and raise the cost of getting them to join the community.

Third, communities with a smaller initial size have less development and higher housing prices. This follows from the fact that $H(H_0)$ is increasing in H_0. This is noteworthy because some of the empirical literature has looked at the relationship between the extent of the current regulatory tax and past development.[20]

5.1 Time consistency

Using standard terminology, the commitment solution $\{H_{t+1}, P_t\}_{t=0}^{\infty}$ is *time consistent* if, for all $t \geq 1$, $\{H_{z+1}, P_z\}_{z=t}^{\infty}$ is an optimal plan for those residents in the community at the beginning of period t, given the initial housing stock H_t. To assess time consistency, we need to understand what optimal plans for future residents look like. The optimal plan for the period t residents will solve the same problem as for the period 0 residents, except that the community's housing stock will be H_t. It will therefore have the same form as that described in Proposition 3. Thus, if $H_t < H^*$ the period t residents will increase the housing stock to $\mathcal{H}(H_t)$ in period t, while if $H_t \geq H^*$ the period t residents will keep the housing stock constant. Given this, we can now establish:

Proposition 4 *Suppose that Assumptions 1 and 2 are satisfied. Then, the initial residents' optimal plan is time consistent if and only if $H_0 \geq H^*$.*

Thus, the commitment solution is time inconsistent when it involves development. Once the population has expanded in the initial period, next period's residents would want to expand it further. Intuitively, the reason they would favor such an expansion and the initial residents would not, is that they do not internalize the negative consequences for the initial period. For example, when new construction is taxed, the initial residents set the tax in all future periods at a level to choke off new construction. Future residents have an incentive to lower this tax to obtain some tax revenue. The initial residents would not countenance such a future reduction because it would adversely impact initial period revenues by encouraging some potential residents to delay entry.

[20] The findings from this work are mixed. See Gyourko and Molloy (2015) for a review. As we will see in the next section, with sequential policy-making, the equilibrium with gradual development predicts the long run community size is independent of its initial size.

Similarly, when new construction is subsidized, the initial residents set the subsidy in all future periods at a level insufficient to attract additional new construction. Future residents have an incentive to increase this subsidy to attract more development. The initial residents would not plan for such a future increase because it would increase the appeal of delaying entry to potential residents and thus increase the costs of subsidization in the initial period.

The proof of Proposition 4 is simple. Recall that if $H_0 < H^*$ the initial residents' optimal plan involves increasing the housing stock to $\mathcal{H}(H_0)$ in period 0. Thereafter, there is no new construction. Now consider the period 1 initial residents. We know that $\mathcal{H}(H_0) < H^*$. Accordingly, they will wish to increase the housing stock to $\mathcal{H}(\mathcal{H}(H_0))$ in period 1. By contrast, if H_0 exceeds H^* the initial residents' optimal plan involves keeping the housing stock constant. The period 1 initial residents therefore face the same trade-off as the initial period residents and want to do the same thing.

6 Sequential policy-making

We now turn to consider the development plans that emerge when policies are chosen sequentially. It is helpful to begin by noting that we can rewrite the equilibrium problem (8) in the following more stream-lined form

$$V(H) = \max_{(H',P)} \left\{ \begin{array}{c} P - \mu\overline{\theta}(1-H') + \mu\beta\left(V(H') - P(H')\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} \\ s.t.\ P = C + \frac{H'}{H}\mathcal{B}(H') + \beta\frac{H'}{H}(P(H') - C)\ \&\ H' \geq H \end{array} \right\}. \tag{18}$$

To see this, note first that the market equilibrium constraint in (8) implies that the surplus from living in the community plus the revenues/costs associated with the tax/subsidy must equal the price of housing less the discounted price of housing next period and the preference of the marginal household. Using this, we can rewrite the objective function as in (18). Second, in the market equilibrium constraint in (8), the price P shows up on the right hand side because it determines the revenues/costs associated with the tax/subsidy. Solving this constraint for P and using the definition of the net private benefit $\mathcal{B}(H)$ in (17) yields the constraint in (18).

6.1 Equilibrium with no development

Given Propositions 3 and 4, it is natural to expect that if $H_0 \geq H^*$, there will exist an equilibrium in which the outcome is the same as in the commitment case. Indeed, it is straightforward to show that if $H_0 \geq H^*$, we can find a solution to problem (18) in which $H'(H) = H$ and $P(H) = C + \mathcal{B}(H)/(1-\beta)$. This yields:[21]

Proposition 5 *Suppose that Assumptions 1 and 2 are satisfied. Then, if $H_0 \geq H^*$, there exists an equilibrium in which no new houses are provided. The price of housing in each period is $C + \mathcal{B}(H_0)/(1-\beta)$ and new construction is taxed.*

6.2 Equilibrium with gradual development

If $H_0 < H^*$, the equilibrium must differ from the initial residents' optimal plan. We begin by looking for an equilibrium in which the housing stock increases gradually over time. This is a natural thing to expect given the nature of the time inconsistency problem. Conveniently, we find such an equilibrium in which the housing rule $H'(H)$ is linear.

To describe this equilibrium, let

$$H^{**} = \frac{\left(\overline{\theta} + S - C(1-\beta) - \underline{u}\right)\xi}{\left(\overline{\theta} - 2s\right)\left(\overline{\theta}(1-\mu) + \xi\right)}, \tag{19}$$

where $\xi = \sqrt{(1-\mu^2\beta)\overline{\theta}\left(\overline{\theta} - 2s\right) + s^2} - s$. This is the steady state to which the housing level converges in our equilibrium. Notice that $H^* \leq H^{**} \leq H^o$ with the equalities holding only when $\mu = 1$. The housing rule in our equilibrium is

$$H'(H) = \begin{cases} \frac{\left(\overline{\theta}+S-C(1-\beta)-\underline{u}\right)\xi}{\left(\overline{\theta}-2s\right)\left(\overline{\theta}+\xi\right)} + \frac{\mu\overline{\theta}}{\overline{\theta}+\xi}H & \text{if } H \in [H_0, H^{**}) \\ H & \text{if } H \in [H^{**}, 1] \end{cases}. \tag{20}$$

When $H < H^{**}$, the housing rule is linear with a positive intercept and a slope less than 1. The definition of H^{**} implies that $H'(H) > H$ for H in this range. Thus, housing is increasing on $[H_0, H^{**})$ and converges asymptotically to H^{**}.

For the associated price rule, define the sequence $\langle H_t(H)\rangle_{t=1}^{\infty}$ inductively as follows: $H_1(H) = H'(H)$ and $H_t(H) = H'(H_{t-1}(H))$ for all $t \geq 2$, where the function $H'(H)$ is as defined in (20). The interpretation is that $H_t(H)$ is the housing level that will prevail at the beginning of the period in t periods time if H is the current housing level and future residents follow the housing rule $H'(H)$. Then the price rule is

$$P(H) = C + \sum_{t=1}^{\infty} \beta^{t-1} \frac{H_t(H)}{H} \mathcal{B}(H_t(H)). \tag{21}$$

[21] The equilibrium described in Proposition 5 is analogous to the stationary equilibrium identified in Ortalo-Magne and Prat (2014).

The value function is given by

$$V(H) = P(H) - \mu\overline{\theta}\left[\sum_{t=1}^{\infty}(\mu\beta)^{t-1}(1 - H_t(H))\right] + \frac{\underline{u}}{1-\beta}. \qquad (22)$$

We can now establish:

Proposition 6 *Suppose that Assumptions 1 and 2 are satisfied and that $H_0 < H^{**}$ where H^{**} is as defined in (19). Then, $\{H'(H), P(H), V(H)\}$ as defined in (20), (21), and (22), is an equilibrium. In this equilibrium, new construction is provided in each period and the housing stock converges asymptotically to H^{**}. If $s < \overline{\theta}(1-\mu)/\left[\mu\sqrt{(1-\beta)} + 1 - \mu\right]$, new construction is taxed in each period. Otherwise, new construction is eventually subsidized, but will initially be taxed if the initial housing stock is sufficiently small. In both cases, the price of housing is decreasing over time.*

In this equilibrium, development is gradual, which creates inefficient delay. This gradual development reflects the underlying time inconsistency problem in the commitment solution. The steady state housing level H^{**} is higher than the housing level that arises in the commitment solution but is still smaller than socially optimal. However, it approaches the social optimum as the probability residents leave the community vanishes.

If the externality is negative, new construction is taxed, but the tax is declining through time, which is what keeps the community growing. In each period, the tax is set so that the benefits of tax revenue raised balance the costs of development in terms of the housing price reduction and lower surplus caused by the negative externality. These benefits and costs anticipate how future residents will respond to the current development.

If the externality is positive, the conditions under which new construction will eventually be subsidized are much less restrictive than in the commitment solution. First, the requirement that s be sufficiently large in Proposition 6 (i.e., exceed $\overline{\theta}(1-\mu)/\left[\mu\sqrt{(1-\beta)} + 1 - \mu\right]$) is much weaker than that in Proposition 4 and can be satisfied when the probability of remaining in the community μ is significantly less than the discount rate β.[22] Second, there is no requirement that the initial housing stock be sufficiently large, since this just determines the behavior of policy during the transition to the steady state. Once enacted, subsidies are increasing through time.

To prove Proposition 6, we begin by considering problem (18) ignoring the second market

[22] For example, if $\beta = 0.95$, $(1-\mu)/\left[\mu\sqrt{(1-\beta)} + 1 - \mu\right]$ is less than $1/2$ for any μ greater than 0.82.

equilibrium constraint that housing cannot decrease. We show that a solution to this unconstrained problem can be used to create a solution to problem (18) if the unconstrained solution satisfies appropriate conditions. We then find a solution to the unconstrained version of problem (18) using the strategy of “guess and verify”. We first conjecture that the housing rule is linear. We then study the unconstrained version of problem (18) under this assumption and find a first order condition which characterizes the optimal housing choice. This first order condition reveals that the optimal housing choice is indeed a linear function of the initial housing level H and this allows us to solve for the intercept and slope of the housing rule. We then go back and verify that this solution satisfies the “appropriate conditions” that allow us to create a solution to problem (18). The solution we create is described in (20), (21), and (22). Finally, we establish the claims about the time path of prices and the use of taxes and subsidies. All this is detailed in the Appendix.

The most interesting feature of this equilibrium is the paths of policy and development it implies. Community development is driven by a gradual drift of policy in a pro-development direction. If a tax on new construction is being employed, it is reduced over time. This reflects the desire of residents to earn the tax revenues that new construction generates.[23] If a subsidy is being employed, it is increased through time. This reflects the desire of residents to obtain the benefits of the positive externality new construction brings. The gradual development that this gives rise to is an appealing and distinctive feature of this equilibrium.

6.3 Equilibrium with stalled development

As we have argued, gradual development is the natural outcome to expect given the nature of the time consistency problem. However, it is not the only possibility. We now identify an equilibrium in which development takes place in the initial period and thereafter is stalled. Further development does not take place because residents anticipate that more development will result in a precipitous fall in the price of housing.

[23] In the case of a development-regulating policy instrument that yielded no revenues, such as a no-fee building permit, the rents from new construction would accrue to developers. A similar gradual increase in building permits would be expected if the community's planning board weighed the developers' rents. This might be the case, for example, if developers could influence the planning board in some way.

For $\widetilde{H}$ in the interval $[H_0, H^{**})$, consider potential solutions of problem (18) of the form

$$H'(H) = \begin{cases} \widetilde{H} & \text{if } H \in [H_0, \widetilde{H}] \\ \frac{\left(\overline{\theta}+S-C(1-\beta)-\underline{u}\right)\xi}{\left(\overline{\theta}-2s\right)\left(\overline{\theta}+\xi\right)} + \frac{\mu\overline{\theta}}{\overline{\theta}+\xi}H & \text{if } H \in (\widetilde{H}, H^{**}) \\ H & \text{if } H \in [H^{**}, 1] \end{cases}, \tag{23}$$

$$P(H) = C + \sum_{t=1}^{\infty} \beta^{t-1} \frac{H_t(H)}{H} \mathcal{B}(H_t(H)), \tag{24}$$

and

$$V(H) = P(H) - \mu\overline{\theta}\left[\sum_{t=1}^{\infty} (\mu\beta)^{t-1} (1 - H_t(H))\right] + \frac{\underline{u}}{1-\beta}. \tag{25}$$

Here, $\langle H_t(H)\rangle_{t=1}^{\infty}$ is defined in the same way as the previous sub-section, except that it uses the housing rule $H'(H)$ defined in (23) (as opposed to (20)). In such a solution, the housing level increases to $\widetilde{H}$ when the current level is below $\widetilde{H}$. When the current level exceeds $\widetilde{H}$, the housing level equals the solution of the previous sub-section. In the equilibrium associated with this solution, the housing level increases to $\widetilde{H}$ in the initial period and then remains there. This is so despite the fact that any small increase of the housing stock beyond $\widetilde{H}$ would cause the housing stock to eventually grow all the way to H^{**}. The price of housing is $C + \widetilde{H}\mathcal{B}(\widetilde{H})/H_0(1-\beta)$ in the initial period and $C + \mathcal{B}(\widetilde{H})/(1-\beta)$ thereafter.

Proposition 7 *Suppose that Assumptions 1 and 2 are satisfied and that $\mu < 1$. Then there exists $\underline{H} < H^*$ such that if $H_0 \in (\underline{H}, H^*)$, there exist equilibria of the form described in (23), (24), and (25) in which the steady state housing level $\widetilde{H}$ is less than the level arising in the commitment solution $\mathcal{H}(H_0)$.*

This proposition establishes that there exist equilibria of the form described in (23), (24), and (25) for a range of initial housing levels. Moreover, in these equilibria, the housing level that emerges is actually less than the level that would be chosen in the commitment solution. The implication of this finding is that sequential policy-making will not necessarily produce more development than would emerge under the commitment solution. Indeed, it may actually enhance the under-supply of housing. The logic underlying this is appealing. In the commitment solution, the initial residents expand housing knowing that they can limit the amount they get. With sequential policy-making, residents do not develop because they cannot control the expansion that will occur after they have developed.[24]

To prove Proposition 7, we first show that for a given $\widetilde{H}$ in the interval $[H_0, H^{**})$, there exists a solution of problem (18) of the form described in (23), (24), and (25) with steady state housing level $\widetilde{H}$ if $\widetilde{H}$ satisfies two conditions. The first guarantees that with a housing stock in the interval $[H_0, \widetilde{H})$, the residents prefer to increase the housing stock to $\widetilde{H}$ than to increase it to some smaller level. The second guarantees that with housing stock in the interval $[H_0, \widetilde{H}]$ the residents prefer $\widetilde{H}$ to any higher level. We then show that there exists $\underline{H} < H^*$ such that if $H_0 \in (\underline{H}, H^*)$ there must be values of $\widetilde{H}$ in the interval $(H_0, \mathcal{H}(H_0))$ satisfying these two conditions. The details can be found in the Appendix.

7 Welfare implications of corrective taxation

Does allowing residents to impose corrective taxation increase or decrease welfare? This involves comparing what happens when residents have access to corrective taxation with free entry. Since sequential policy-making is the most realistic assumption, the equilibria that arise in this case are a more natural point of comparison than the commitment solution. Nonetheless, Proposition 5 tells us that if $H_0 \geq H^*$, there will exist an equilibrium with sequential policy-making in which the outcome is the same as in the commitment solution. Moreover, Proposition 7 tells us that, for a range of initial housing levels, there exist stalled equilibria which have the same form as the commitment solution, but produce a lower housing level. Given that the commitment solution under-supplies housing, these equilibria produce a lower level of welfare. Accordingly, understanding how the commitment solution compares with free entry is informative for the question at hand and, since it is more tractable, we begin with it.

7.1 The commitment solution versus free entry

The commitment solution and free entry equilibrium have the same timing in the sense that all new construction occurs in the initial period. For a welfare analysis, we just need to compare the amount of new construction in the two solutions.

[24] The stalled equilibrium relates to a conjecture in Ortalo-Magne and Prat (2014). They note: "without stationarity, there could be equilibria with an extremely small city. Intuitively, even a small size increase today could create the "expectation" of large increases in the future. Any deviation today would trigger a collapse in house prices. Hence, current generations do not modify the city size, even when it is extremely low." (p.163).

If the externality is positive, the free entry housing level is smaller than the social optimum. Since the housing level in the commitment solution is also smaller than the social optimum, the only question is which is larger. When $H_0 \geq H^*$, free entry will be better because the housing level is just H_0 in the commitment solution and, by Assumption 2, this is smaller than H^e. When $H_0 < H^*$, the housing level in the commitment solution will be larger than the free entry level if and only if new construction is subsidized. Proposition 3 tells us that new construction is subsidized only when the externality and initial housing stock are above some bounds. These then are the conditions under which welfare is higher under the commitment solution. As already discussed, these conditions are restrictive.

If the externality is negative, the free entry housing level exceeds the social optimum, while the housing level in the commitment solution is smaller than the social optimum. The question is therefore whether the social cost of excessive new construction exceeds that of insufficient new construction. Intuitively, one might guess that if the externality is small in absolute value, free entry will dominate, but the commitment solution might do better with a large externality. As the next Proposition shows, this is approximately correct, but the initial housing stock still plays a key role, since it determines the level of taxation in the commitment solution.

Proposition 8 *Suppose that Assumptions 1 and 2 are satisfied.*

(i) Suppose that $s > 0$. Then, if $H_0 < H^*$, welfare is higher with free entry than under the commitment solution if

$$s \leq \overline{\theta}\frac{1-\mu}{1-\mu\beta}. \tag{26}$$

If (26) is not satisfied, welfare is higher under the commitment solution if and only if $H_0 > \frac{\overline{\theta}-2s}{\overline{\theta}}\frac{1-\mu\beta}{\mu(1-\beta)}H^o$. If $H_0 \geq H^*$, welfare is higher with free entry.

(ii) Suppose that $s < 0$. Then, if $H_0 < H^*$, welfare is higher with free entry if

$$s > -\overline{\theta}\frac{\sqrt{1+8\frac{1-\mu}{1-\mu\beta}}-1}{4}. \tag{27}$$

If (27) is not satisfied, welfare is higher under the commitment solution if and only if $H_0 > \frac{\overline{\theta}+2s}{\overline{\theta}}\frac{1-\mu\beta}{\mu(1-\beta)}H^o$. If $H_0 \geq H^*$ and (27) is satisfied, welfare is higher under free entry if and only if $H_0 < \frac{\overline{\theta}}{\overline{\theta}-s}H^o$. If (27) is not satisfied, welfare is higher under the commitment solution.

The proof of this Proposition can be found in the on-line Appendix. For any given (H_0, s) pair, it tells us whether welfare will be higher with the commitment solution or free entry. Figure

1 illustrates the Proposition for a particular parameterization. The housing level H is measured on the horizontal axis and the externality s is measured on the vertical. The parameters of the model other than H_0 and s are fixed at the following values[25]

Parameter	$\bar{\theta}$	β	μ	C	S	$\underline{u}$
Value	1	.95	.95	17	0	0

The blue (solid), black (dotted), and green (dashed) lines depict, respectively, the housing levels H^o, H^e, and H^* associated with any given externality level s. Assumptions 1(ii) and 2 require that the initial housing stock H_0 is smaller than both H^o and H^e. The gray (light) and orange (dark) shaded areas represent (H_0, s) pairs which satisfy our Assumptions[26] and the orange shaded area represents pairs for which welfare is higher with the commitment solution. The Figure indicates that, when the externality is positive, welfare is always higher with free entry. This follows immediately from the fact that μ is equal to β. When the externality is negative, it is possible that the commitment solution dominates even for very small s if $H_0 \geq H^*$. However, if $H_0 < H^*$, the externality must be below (27) and the initial housing level must be above the bound given in the Proposition.

The Figure illustrates the limited set of initial conditions for which the commitment solution dominates. A more optimistic picture can only be obtained by increasing μ closer to 1. With μ equal to .99, for example, the commitment solution dominates for a larger slice of the space with negative externalities and also for some of the space with positive externalities (see Figure 6 in the on-line Appendix). However, whatever value of μ chosen, there remains an asymmetry between positive and negative externalities, in that the commitment solution has a greater advantage when the externality is negative. This reflects the fact that dealing with a negative externality requires taxing, while tackling a positive externality requires subsidization. Taxing is more attractive to the initial residents for obvious reasons. Furthermore, it is always the case that, for the commitment solution to have an advantage, the initial housing stock must be relatively large.

[25] It is easy to show that s, C, $\bar{\theta}$, $\underline{u}$ and S do not play an independent role in determining the behavior of the model. The latter is determined by $\{\beta, \mu, \hat{s}, \hat{u}\}$, where $\hat{s} \equiv s/\bar{\theta}$, and $\hat{u} \equiv -\frac{S-C(1-\beta)-\underline{u}}{\theta}$. Hence, without loss of generality, we set $\bar{\theta} = 1$ and $\underline{u} = S = 0$.

[26] The upper bound on s is determined by Assumption 1. For symmetry, we choose the lower bound to have the same absolute value as the upper bound.

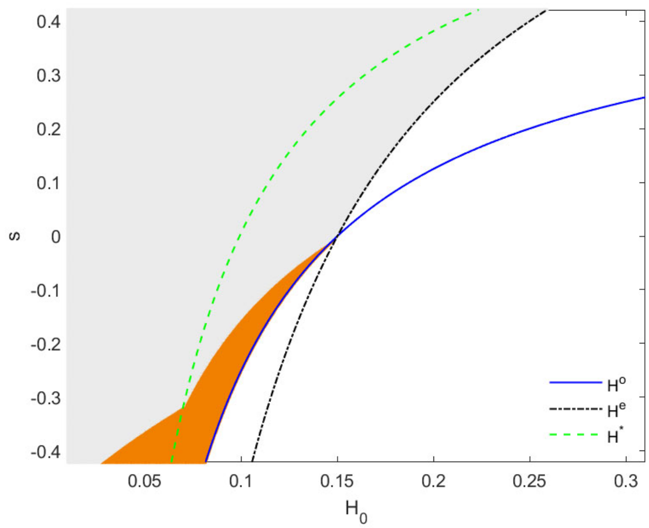

Figure 1: Commitment versus Free Entry

7.2 Sequential policy-making versus free entry

We begin with the equilibrium with gradual development. An immediate question is how welfare in this equilibrium compares with welfare under the commitment solution. This equilibrium generates a steady state housing level exceeding that with commitment. Because this level remains bounded by the social optimum, this means that the eventual level of welfare must be higher under this equilibrium. On the other hand, development is gradual, so the housing level could be below that with commitment for some period of time.[27] Given these two off-setting forces, it is not immediate that welfare will be higher under the equilibrium with gradual development. Because of the complexity of welfare under this equilibrium, an analytical understanding of this comparison is difficult. Nonetheless, it is possible to investigate the issue numerically. We have found no parameter values under which welfare in the equilibrium with gradual development is not at least as high as commitment welfare.[28] Thus, the benefits of a higher steady state housing level appear

[27] Note that the intercept of the equilibrium housing rule $H'(H)$ as defined in (20) is smaller than that of the commitment solution $\mathcal{H}(H)$ as defined in (16).

[28] We performed these calculations for more than 19 million sets of parameter values. Specifically, defining $\hat{u}$ and $\hat{s}$ as in footnote #25, we allowed μ, β, and $\hat{u}$ to take values from .01 to .99 with an increment of .01, and $\hat{s}$ to take values from -.99 to .99 (with an increment of .01) of its respective upper bound. For each set of parameter values, we considered all H_0 on a 1000 point uniform grid on [0,1] that satisfy our Assumptions.

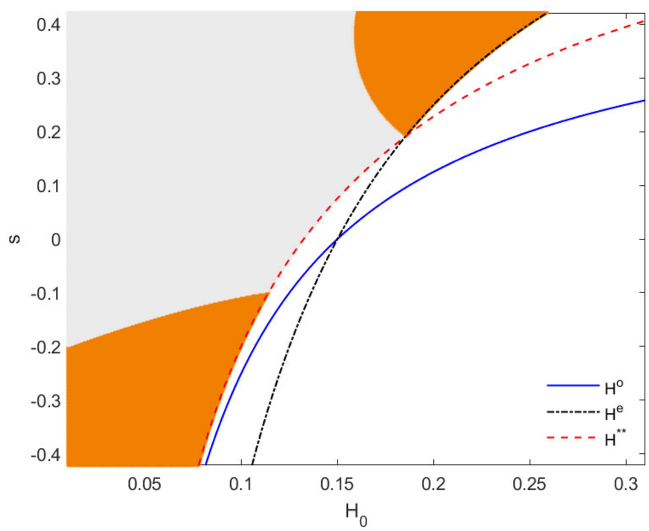

Figure 2: Gradual Equilibrium versus Free Entry

more than sufficient to offset the costs of gradual development.

The fact that the equilibrium with gradual development dominates the commitment solution, suggests that it will dominate free entry for a more significant part of the parameter space. This is illustrated in Figure 2 which uses the same underlying parameters as Figure 1.[29] The blue (solid), black (dotted), and red (dashed) lines depict, respectively, the housing levels H^o, H^e, and H^{**} associated with any given externality level s. The gray (light) and orange (dark) shaded areas represent (H_0, s) pairs which satisfy our Assumptions and for which a gradual equilibrium exists (which requires $H_0 < H^{**}$). The orange shaded areas represent (H_0, s) pairs under which welfare is higher under the equilibrium with gradual development than with free entry. The ratio of these areas to the total shaded area (i.e., gray plus orange) is evidently significantly larger than the same ratio in Figure 1, illustrating the potential benefits from sequential policy-making.[30] Nonetheless, it is still the case that for small values of the externality, free entry dominates. Moreover, when the externality is positive and subsidization is necessary, free entry is dominated only when the initial housing stock is sufficiently large. This reflects the fact that, with a low initial housing

[29] There is nothing special about these parameter values. A similar picture emerges with other parameter choices.

[30] Note that the size of the total shaded area (gray plus orange) in Figure 2 is smaller than that in Figure 1 because it excludes points for which a gradual equilibrium does not exist (i.e, $H_0 \geq H^{**}$).

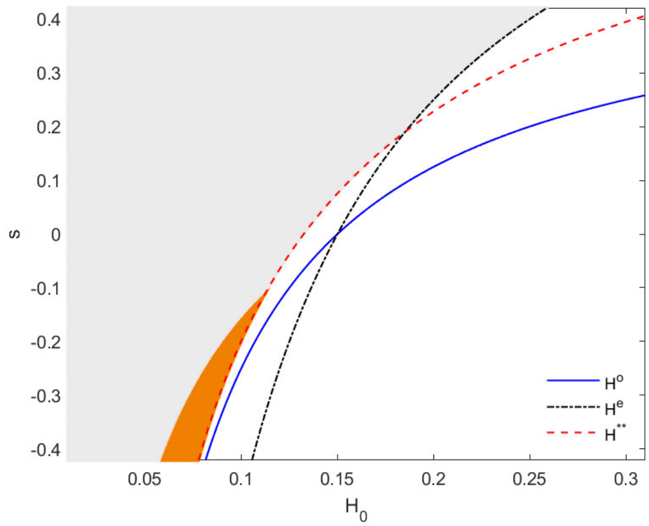

Figure 3: Worst Stalled Equilibrium versus Free Entry

stock, the community taxes new construction for a number of periods in the transition to the steady state. Thus, while it does not alter the long run size of the community, the initial housing stock still plays an important role in determining welfare.

Turning to the stalled equilibrium, there are multiple equilibria of this form each characterized by an associated steady state housing level. Like the commitment solution, these equilibria have the same timing as the free entry equilibrium in the sense that all new construction occurs in the initial period. Thus, for a welfare analysis, we just need to compare the amount of new construction in the two solutions. Obviously, the results will depend on which equilibrium is considered. Equilibria in which development stalls after a larger burst of new construction in the initial period will perform better. For our current purposes, we are interested in worst case scenarios. In this regard, the important point to note is that Proposition 7 tells us that for a range of initial housing levels there exist stalled equilibria in which the steady state housing level is less than that associated with the commitment solution. Thus, the welfare performance of such stalled equilibria will be even worse. This is illustrated in Figure 3. The gray and orange shaded areas in Figure 3 represent (H_0, s) pairs which satisfy Assumptions 1 and 2 and for which a stalled equilibrium exists (which again requires $H_0 < H^{**}$). The orange shaded area represents

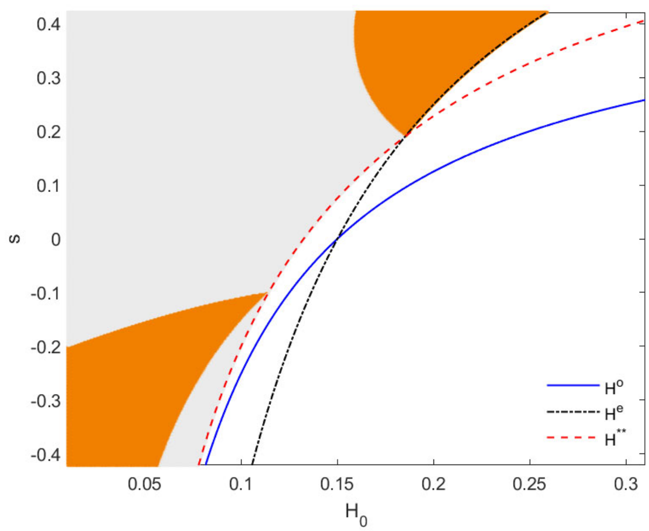

Figure 4: Region of Ambiguous Effect of Corrective Taxation

combinations for which the worst stalled equilibrium (i.e., the one with the smallest steady state housing level $\widetilde{H}$) dominates free entry. Comparing Figures 1 and 3, it is clear that the worst stalled equilibrium under-performs the commitment solution.

Pulling together the information from Figures 2 and 3, we see that, for a large range of (H_0, s) pairs, there exists an equilibrium - the worst stalled equilibrium - which is worse than free entry and an equilibrium - the one with gradual development - that is better. The orange shaded area in Figure 4 illustrates this set of points. For this set, the answer to the question "does allowing residents to impose corrective taxation increase welfare?" is "maybe but maybe not".

This message of ambiguity not withstanding, we can be more confident that allowing residents to impose corrective taxation will reduce welfare when externalities are small. In almost all such cases, free entry dominates the outcome with sequential policy-making whichever equilibrium arises. This is illustrated in Figure 5. The gray and orange shaded areas represent (H_0, s) pairs which satisfy our Assumptions and the orange shaded areas represent pairs for which the best of the equilibria with sequential policy-making identified in Section 6 dominates free entry. When $H_0 \geq H^{**}$, this best equilibrium is just the equilibrium with no development identified in Proposition 5. When $H_0 < H^{**}$, this best equilibrium is either the gradual equilibrium or the best stalled

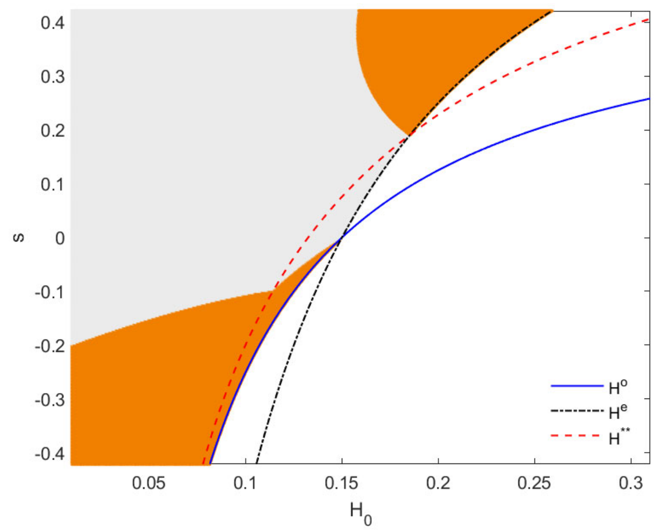

Figure 5: Best Sequential Policy-Making Equilibrium versus Free Entry

equilibrium.[31] For externalities less than, say, 0.1 in absolute value, free entry dominates for all but a negligible set of points. Given that free entry generates the optimal development plan when there are no externalities, this is not a surprising finding. However, it does underscore the point that, while the distortions associated with free entry vanish as externalities become small, those arising from residents choosing corrective taxes do not.

8 Conclusion

This paper analyzes community development with externalities and corrective taxation under the assumption that policies are determined by resident homeowners. The choice of corrective taxation is distorted by the desire of residents to raise the value of their homes and to obtain the revenues from taxation or, if subsidies are warranted, to avoid paying their cost. In a world in which the initial residents can commit the community to future policies, these forces play out

[31] The best stalled equilibrium is that associated with the highest possible steady state housing level $\widetilde{H}$. This $\widetilde{H}$ has the property that the initial residents are just willing to choose it knowing that, if they choose a lower level, next period's residents would increase housing to $\widetilde{H}$ next period. This $\widetilde{H}$ will exceed the commitment level $\mathcal{H}(H_0)$ which is why it can sometimes generate a higher welfare level than the gradual equilibrium. The reason it can exceed the commitment level is that the fact that next period's residents will increase to $\widetilde{H}$ in any case, induces the initial period residents to focus on the short run benefits of more development since the long run costs cannot be avoided. Figure 7 in the on-line Appendix shows that the best stalled equilibrium dominates the gradual equilibrium when H_0 is sufficiently close to H^{**}. Intuitively, such conditions permit a higher maximum $\widetilde{H}$ which allows the benefits of no delay to more than offset the costs of a lower steady state level.

in a relatively simple way. They lead residents to distort corrective taxation in the direction of restricting development. The distortion is larger when the initial size of the community is smaller, because this raises per capita tax revenues and subsidy costs. All this means that if externalities are small, social welfare will be higher if residents do not have access to corrective taxation. When externalities are large, corrective taxation can be beneficial, but this will depend on whether the externality is positive or negative. In the former case, the conditions for corrective taxation to be beneficial are restrictive. In the latter, they are weaker, but still require that the initial size of the community be sufficiently large.

When the commitment solution involves development, it is time inconsistent in the sense that future residents want to develop further. This suggests that, in the more realistic case of sequential policy-making, the problems of restricting development will be attenuated. There is some truth to this, in the sense that there exists an equilibrium with gradual development in which development expands well beyond the commitment level. Indeed, under some conditions, development in the long run can almost reach the optimal level. While development proceeds too slowly in this equilibrium, welfare is higher than in the commitment solution. Even here, if externalities are small, social welfare will typically be higher if residents do not have access to corrective taxation. Moreover, when the externality is large, for corrective taxation to be beneficial still requires that the initial size of the community not be too small if the externality is positive.

Complicating matters, with sequential policy-making there also exist equilibria with stalled development. In these equilibria, development takes place only in the initial period and then stops. Development is stalled by the rational fear of residents that more development will result in a precipitous fall in the price of housing. Most significantly, the extent of development in these equilibria can be strictly less than in the commitment solution. Given that the commitment solution under-supplies housing, the welfare performance of these equilibria can be even worse. All this means that for large externalities, it is often possible to find equilibria with sequential policy-making in which welfare is higher and lower than under free entry.

These results make a tenuous case for allowing communities to control corrective taxation in the presence of development externalities. As noted in the introduction, these externalities can be tackled with a variety of different policy instruments. The forces that distort the choice of corrective taxation, will also shape the choice of alternative instruments. Indeed, the restrictions associated with these instruments may mean they perform even worse. For example, if zoning does not allow a community to raise revenue from creating developable land and developers have no influence over the zoning authority, development will be evenly more seriously restricted than with taxes.[32] On the other hand, the public wealth accumulation mechanism identified in Barseghyan and Coate (2018) may prove superior to corrective taxation in settings with a positive externality. Understanding the performance of different instruments in various settings, taking account of their political determination, seems a worthwhile agenda for further research. So to does understanding how development-regulating policies might be set by higher levels of government.

[32] This is an argument made in the literature on development impact fees. These fees are analogous to taxes on new construction, although the revenues must be earmarked for capital expenditures related to infrastructure expansions necessary to accomodate development. A number of authors have noted that such fees may permit more development than zoning. See Burge (2010) for a good overview of the discussion.

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9 Appendix

9.1 Proof of Proposition 3

We approach problem (14) through a process of successive simplification. Our first simplifying observation concerns the objective function.

Fact 1. *Suppose that the sequence of policies* $\{H_{t+1}, P_t\}_{t=0}^{\infty}$ *satisfies in each period* t *the market equilibrium condition*

$$P_t = (1 - H_{t+1})\overline{\theta} + S(H_{t+1}) + \frac{(P_t - C)(H_{t+1} - H_t)}{H_{t+1}} + \beta P_{t+1} - \underline{u} \tag{28}$$

and the transversality condition that $\lim_{t\to\infty} \beta^t P_t = 0$*. Then, the initial period residents' objective function satisfies*

$$\begin{aligned} &\textstyle\sum_{t=0}^{\infty} (\mu\beta)^t \left\{(1-\mu)\left[P_t + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H_{t+1}) + \frac{(P_t - C)(H_{t+1}-H_t)}{H_{t+1}}\right]\right\} \\ &\qquad = P_0 - \textstyle\sum_{t=0}^{\infty} (\mu\beta)^t \mu(1 - H_{t+1})\overline{\theta} + \frac{\underline{u}}{1-\beta}. \end{aligned} \tag{29}$$

Proof of Fact 1. From the market equilibrium condition, we have that

$$S(H_{t+1}) + \frac{(P_t - C)(H_{t+1} - H_t)}{H_{t+1}} = P_t - (1 - H_{t+1})\overline{\theta} - \beta P_{t+1} + \underline{u}.$$

Thus, for any period $t \geq 0$, we have that

$$\begin{aligned} & \sum_{z=0}^{t} (\mu\beta)^z \left\{(1-\mu)\left[P_z + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H_{z+1}) + \frac{(P_z - C)(H_{z+1} - H_z)}{H_{z+1}}\right]\right\} \\ = & \sum_{z=0}^{t} (\mu\beta)^z \left\{(1-\mu)\left[P_z + \frac{\underline{u}}{1-\beta}\right] + \mu\left[P_z - (1 - H_{z+1})\overline{\theta} - \beta P_{z+1} + \underline{u}\right]\right\}. \end{aligned}$$

Expanding the right hand side, we have that

$$\begin{aligned} & \sum_{z=0}^{t} (\mu\beta)^z \left\{(1-\mu)\left[P_z + \frac{\underline{u}}{1-\beta}\right] + \mu\left[P_z - (1 - H_{z+1})\overline{\theta} - \beta P_{z+1} + \underline{u}\right]\right\} \\ = & (1-\mu)\left[P_0 + \frac{\underline{u}}{1-\beta}\right] + \mu\left[P_0 - (1 - H_1)\overline{\theta} - \beta P_1 + \underline{u}\right] \\ & +\mu\beta\left\{(1-\mu)\left[P_1 + \frac{\underline{u}}{1-\beta}\right] + \mu\left[P_1 - (1 - H_2)\overline{\theta} - \beta P_2 + \underline{u}\right]\right\} \\ & +(\mu\beta)^2\left\{(1-\mu)\left[P_2 + \frac{\underline{u}}{1-\beta}\right] + \mu\left[P_2 - (1 - H_3)\overline{\theta} - \beta P_3 + \underline{u}\right]\right\} \\ & +\ldots.. + (\mu\beta)^t\left\{(1-\mu)\left[P_t + \frac{\underline{u}}{1-\beta}\right] + \mu\left[P_t - (1 - H_{t+1})\overline{\theta} - \beta P_{t+1} + \underline{u}\right]\right\}. \end{aligned}$$

Note that the P_1 term in the first line on the right hand side of the equality cancels with that in the second line. Similarly, the P_2 term in the second line cancels with the P_2 term in the third line, etc. Thus, we have

$$\begin{aligned} & \sum_{z=0}^{t} (\mu\beta)^z \left\{ (1-\mu) \left[P_z + \frac{u}{1-\beta} \right] + \mu \left[P_z - (1-H_{z+1})\overline{\theta} - \beta P_{z+1} + \underline{u} \right] \right\} \\ = \; & P_0 + \sum_{z=0}^{t} (\mu\beta)^z \left\{ (1-\mu) \frac{u}{1-\beta} + \mu \left[-(1-H_{z+1})\overline{\theta} + \underline{u} \right] \right\} - (\mu\beta)^{t+1} P_{t+1}. \end{aligned}$$

The transversality condition implies that $\lim_{t\to\infty} (\mu\beta)^{t+1} P_{t+1} = 0$. Thus,

$$\begin{aligned} & \sum_{t=0}^{\infty} (\mu\beta)^t \left\{ (1-\mu) \left[P_t + \frac{u}{1-\beta} \right] + \mu \left[P_t - (1-H_{t+1})\overline{\theta} - \beta P_{t+1} + \underline{u} \right] \right\} \\ = \; & P_0 - \sum_{t=0}^{\infty} (\mu\beta)^t \mu (1-H_{t+1})\overline{\theta} + \frac{u}{1-\beta}. \end{aligned}$$

■

This result reveals that housing prices have a direct impact on the objective function only in the initial period and that, all else equal, the residents prefer to have future housing stocks as high as possible. We can therefore write the problem as:

$$\max_{\{H_{t+1},P_t\}_{t=0}^{\infty}} \left\{ \begin{array}{c} P_0 - \sum_{t=0}^{\infty} (\mu\beta)^t \mu (1-H_{t+1})\overline{\theta} + \frac{u}{1-\beta} \\ s.t. \text{ for all } t \geq 0 \\ P_t = (1-H_{t+1})\overline{\theta} + S(H_{t+1}) + \frac{(P_t - C)(H_{t+1}-H_t)}{H_{t+1}} + \beta P_{t+1} - \underline{u} \;\&\; H_{t+1} \geq H_t \end{array} \right\}. \tag{30}$$

Our next simplifying result provides a convenient expression for the price P_0.

Fact 2. *Suppose that the sequence of policies $\{H_{t+1}, P_t\}_{t=0}^{\infty}$ satisfies in each period t the market equilibrium condition (28) and the transversality condition that $\lim_{t\to\infty} \beta^t P_t = 0$. Then,*

$$P_0 = C + \sum_{t=0}^{\infty} \beta^t \frac{H_{t+1}\mathcal{B}(H_{t+1})}{H_0}. \tag{31}$$

Proof of Fact 2. Note that P_t appears on the left and right hand side of the market equilibrium constraint (28). Solving the constraint for P_t reveals that

$$P_t = C + \frac{H_{t+1}\mathcal{B}(H_{t+1})}{H_t} + \beta \frac{H_{t+1}\left[P_{t+1} - C\right]}{H_t}.$$

It follows that if all the future market equilibrium constraints are satisfied, it must be the case that

$$\begin{aligned} P_0 &= C + \frac{H_1 \mathcal{B}(H_1)}{H_0} + \beta \frac{H_1 \left[P_1 - C\right]}{H_0} \\ &= C + \frac{H_1 \mathcal{B}(H_1)}{H_0} + \beta \frac{H_2 \mathcal{B}(H_2)}{H_0} + \beta^2 \frac{H_2 \left[P_2 - C\right]}{H_0}. \end{aligned}$$

Generalizing this logic, for all $t \geq 1$ we can write:

$$P_0 = C + \sum_{z=0}^{t-1} \beta^z \frac{H_{z+1} \mathcal{B}(H_{z+1})}{H_0} + \beta^t \frac{H_t \left[P_t - C\right]}{H_0}.$$

The transversality condition implies that $\lim_{t \to \infty} \beta^t H_t \left[P_t - C\right] = 0$ and thus, we have that

$$P_0 = C + \sum_{t=0}^{\infty} \beta^t \frac{H_{t+1} \mathcal{B}(H_{t+1})}{H_0}.$$

■

Using Fact 2, we can write the problem as

$$\max_{\{H_{t+1}\}_{t=0}^{\infty}} \left\{ \begin{array}{c} C + \sum_{t=0}^{\infty} \beta^t \frac{H_{t+1} \mathcal{B}(H_{t+1})}{H_0} - \sum_{t=0}^{\infty} (\mu\beta)^t \mu(1 - H_{t+1})\overline{\theta} + \frac{u}{1-\beta} \\ s.t.\ H_{t+1} \geq H_t \text{ for all } t \geq 0 \end{array} \right\}.$$

This substantially simplifies matters, since all the prices are removed. Our next simplifying result tells us that all new construction occurs in the initial period.

Fact 3 *Let $\{H_{t+1}\}_{t=0}^{\infty}$ solve the initial period residents' problem. Then, for all $t \geq 1$*

$$H_{t+1} = H_1.$$

Proof of Fact 3 Suppose the contrary. Then there exist some period $t \geq 1$ such that $H_{t+1} > H_t$. It follows that we can increase the value of H_t marginally without violating the constraints. The change in the objective function is

$$\beta^{t-1} \left[\frac{H_t \mathcal{B}'(H_t) + \mathcal{B}(H_t)}{H_0}\right] + \mu\overline{\theta} \left(\mu\beta\right)^{t-1}.$$

It follows that

$$\frac{H_t \mathcal{B}'(H_t) + \mathcal{B}(H_t)}{H_0} + \mu^t \overline{\theta} \leq 0.$$

But we can also reduce the value of H_{t+1} marginally without violating the constraints. The change in the objective function resulting from such a change is

$$-\left(\beta^t \left[\frac{H_{t+1} \mathcal{B}'(H_{t+1}) + \mathcal{B}(H_{t+1})}{H_0}\right] + \mu\overline{\theta} \left(\mu\beta\right)^t\right).$$

It follows that

$$\frac{H_{t+1}\mathcal{B}'(H_{t+1})+\mathcal{B}(H_{t+1})}{H_0}+\mu^{t+1}\overline{\theta}\geq 0.$$

Combining these two inequalities, we find that

$$H_{t+1}\mathcal{B}'(H_{t+1})+\mathcal{B}(H_{t+1})\geq H_t\mathcal{B}'(H_t)+\mathcal{B}(H_t).$$

However, we have that

$$\begin{aligned}\frac{d\left(H\mathcal{B}'(H)+\mathcal{B}(H)\right)}{dH} &= H\mathcal{B}''(H)+2\mathcal{B}'(H)\\ &= -2\left(\overline{\theta}-s\right)<0,\end{aligned}$$

where the inequality follows from Assumption 1. Since $H_{t+1}>H_t$, this implies that

$$H_{t+1}\mathcal{B}'(H_{t+1})+\mathcal{B}(H_{t+1})<H_t\mathcal{B}'(H_t)+\mathcal{B}(H_t),$$

which is a contradiction. ■

This result allows us to reduce the initial period residents' problem to the following very simple problem involving only one choice variable

$$\max_{\{H_1\}}\left\{\begin{array}{c} C+\frac{H_1\mathcal{B}(H_1)}{H_0(1-\beta)}-\frac{\mu\overline{\theta}(1-H_1)}{1-\mu\beta}+\frac{u}{1-\beta}\\ s.t.\ H_1\geq H_0\end{array}\right\}. \tag{32}$$

Let H^* denote the housing level defined in (15) the text and let $\mathcal{H}(H)$ be the function defined in (16). Then, we have the following result.

Fact 4 *The optimal level of housing for the initial period residents is H_0 if $H_0\geq H^*$. Otherwise, it is equal to $\mathcal{H}(H_0)$.*

Proof of Fact 4 The first derivative of the objective function is

$$\frac{H_1\mathcal{B}'(H_1)+\mathcal{B}(H_1)}{H_0(1-\beta)}+\frac{\mu\overline{\theta}}{1-\mu\beta}.$$

The second derivative is

$$\frac{\frac{d\left(H_1\mathcal{B}'(H_1)+\mathcal{B}(H_1)\right)}{dH}}{H_0(1-\beta)}=-\frac{2\left(\overline{\theta}-s\right)}{H_0(1-\beta)}<0,$$

implying that the objective function is strictly concave. It follows that if

$$\frac{H_0\mathcal{B}'(H_0)+\mathcal{B}(H_0)}{H_0(1-\beta)}+\frac{\mu\overline{\theta}}{1-\mu\beta}\leq 0,$$

then the optimal housing level is H_0. Since

$$H\mathcal{B}'(H) + \mathcal{B}(H) = (1-2H)\overline{\theta} + S + 2sH - C(1-\beta) - \underline{u},$$

this will be the case if

$$(1-2H_0)\overline{\theta} + S + 2sH_0 - C(1-\beta) - \underline{u} \leq -H_0\frac{\mu(1-\beta)}{1-\mu\beta}\overline{\theta},$$

or, equivalently, if

$$H_0 \geq \frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{\overline{\theta}(1+\frac{1-\mu}{1-\mu\beta}) - 2s} = H^*.$$

If $H_0 < H^*$ then the first order condition is

$$\frac{H_1\mathcal{B}'(H_1) + \mathcal{B}(H_1)}{H_0(1-\beta)} + \frac{\mu\overline{\theta}}{1-\mu\beta} = 0. \tag{33}$$

Rearranging this, we see that the optimal housing level satisfies the equation

$$(1-H_1)\overline{\theta} + S + 2sH_1 - \underline{u} - C(1-\beta) = \overline{\theta}\left(H_1 - H_0\frac{\mu(1-\beta)}{1-\mu\beta}\right), \tag{34}$$

which implies that it equals $\mathcal{H}(H_0)$. ■

We have therefore established that the solution to the initial period residents' problem is as follows. If $H_0 < H^*$, $\mathcal{H}(H_0) - H_0$ new houses are provided in the initial period. Thereafter, no more housing is provided. From (31), the price of housing is $C + \mathcal{H}(H_0)\mathcal{B}(\mathcal{H}(H_0))/H_0(1-\beta)$ in the initial period and $C + \mathcal{B}(\mathcal{H}(H_0))/(1-\beta)$ thereafter. If $H_0 \geq H^*$, no new houses are provided. The price of housing in all periods is $C + \mathcal{B}(H_0)/(1-\beta)$.

To understand how the conclusions concerning the taxation or subsidization of new construction follow from the price expressions, note that at the free entry housing level H^e, $\mathcal{B}(H^e) = 0$. Since $\mathcal{B}(H)$ is decreasing in H, it follows that when $H_0 < H^*$ new construction will be taxed if $\mathcal{H}(H_0) < H^e$ and subsidized if $\mathcal{H}(H_0) > H^e$. When $s < 0$, new construction will be taxed because $H^e > H^o > \mathcal{H}(H_0)$. When $s > 0$, new construction may be subsidized, but it requires that H_0 be sufficiently large. More specifically, it is necessary that $H_0 \in (\frac{\overline{\theta}-s}{\overline{\theta}}\frac{1-\mu\beta}{\mu(1-\beta)}H^e, H^e)$. The upper bound on this interval comes from Assumption 2, while the lower bound comes from using (16) to compute the condition under which $\mathcal{H}(H_0) > H^e$. This interval is non-empty if and only if $s > \overline{\theta}(1-\mu)/(1-\mu\beta)$. It follows that when $s < \overline{\theta}(1-\mu)/(1-\mu\beta)$, new construction will be taxed whatever the initial housing level. However, when $s < \overline{\theta}(1-\mu)/(1-\mu\beta)$, new construction will be subsidized if

$$H_0 > \frac{\overline{\theta} - s}{\overline{\theta}} \frac{1 - \mu\beta}{\mu(1-\beta)} H^e = \frac{\overline{\theta} - 2s}{\overline{\theta}} \frac{1 - \mu\beta}{\mu(1-\beta)} H^o.$$

When $H_0 > H^*$ new construction will be taxed since $H_0 < H^e$ by Assumption 2.

9.2 Proof of Proposition 6

9.2.1 A preliminary result

We begin by considering a relaxed version of problem (18) which ignores the constraint that housing cannot decrease. The following result establishes that a solution to this unconstrained problem can be used to create a solution to problem (18) if the unconstrained solution satisfies certain conditions. The proof can be found in the on-line Appendix.

Lemma 1 *Let $\{H_u(H), P_u(H), V_u(H)\}$ be a solution to the problem*

$$V(H) = \max_{(H',P)} \left\{ \begin{array}{c} P - \mu\overline{\theta}(1 - H') + \mu\beta\left(V(H') - P(H')\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} \\ s.t.\ P = C + \frac{H'}{H}\mathcal{B}(H') + \beta\frac{H'}{H}(P(H') - C) \end{array} \right\}. \tag{35}$$

Suppose further that the housing rule $H_u(H)$ is increasing on the interval $[H_0, 1]$, that $H_u(H_0) > H_0$, that $H_u(1) < 1$, and that there exists a unique housing level $\widetilde{H} \geq H^*$ such that $H_u(\widetilde{H}) = \widetilde{H}$. Then, there exists a solution of problem (18) in which

$$H'(H) = \left\{ \begin{array}{c} H_u'(H) \text{ if } H \in [H_0, \widetilde{H}) \\ \\ H \text{ if } H \in [\widetilde{H}, 1] \end{array} \right., \tag{36}$$

$$P(H) = \left\{ \begin{array}{c} P_u(H) \text{ if } H \in [H_0, \widetilde{H}) \\ \\ C + \frac{\mathcal{B}(H)}{1-\beta} \text{ if } H \in [\widetilde{H}, 1] \end{array} \right., \tag{37}$$

and

$$V(H) = \left\{ \begin{array}{c} V_u(H) \text{ if } H \in [H_0, \widetilde{H}) \\ \\ C + \frac{\mathcal{B}(H)}{1-\beta} - \frac{\mu\overline{\theta}}{1-\mu\beta}(1 - H) + \frac{\underline{u}}{1-\beta} \text{ if } H \in [\widetilde{H}, 1] \end{array} \right.. \tag{38}$$

9.2.2 Solving the unconstrained problem

We now turn to solving the unconstrained problem (35). For any candidate housing rule $H_u(H)$, define the sequence $\langle H_{ut}(H)\rangle_{t=1}^{\infty}$ inductively as follows: $H_{u1}(H) = H_u(H)$, $H_{ut}(H) = H_u(H_{ut-1}(H))$

for all $t \geq 2$. The interpretation is that $H_{ut}(H)$ is the housing level that will prevail in t periods time if H is the housing level selected this period and future residents follow the housing rule $H_u(H)$. Using this notation, we can write the unconstrained problem (18) as follows:

$$\max_{(H',P)} \left\{ \begin{array}{c} P - \mu\overline{\theta}\left[1 - H' + \sum_{t=1}^{\infty} (\mu\beta)^t (1 - H_{ut}(H'))\right] + \frac{u}{1-\beta} \\ s.t.\ P = C + \frac{H'}{H}\mathcal{B}(H') + \sum_{t=1}^{\infty} \beta^t \frac{H_{ut}(H')}{H}\mathcal{B}(H_{ut}(H')) \end{array} \right\}. \tag{39}$$

This way of writing the problem is analogous to our treatment of the commitment problem in the proof of Proposition 3 (see (30)). The objective function reveals that all that matters to the residents is the current housing price and a $\mu\beta$ discounted sum of future housing levels. The expression for the price (which is similar to (31)) reveals that it can be written as the cost of construction plus a term which consists of the discounted (and population weighted) sum of the net private benefit that would be obtained by the marginal household if residents did not have access to a new construction tax or subsidy. The key difference between this and problem (30) is that in the latter the residents' directly choose each period's housing level. In the former, the residents choose only this period's housing and this indirectly determines future housing levels through its impact on the sequence $\langle H_{ut}(H')\rangle_{t=1}^{\infty}$.

Substituting the price into the objective function and ignoring constant terms we can reduce problem (39) to

$$\max_{H'} \frac{H'}{H}\mathcal{B}(H') + \sum_{t=1}^{\infty} \beta^t \frac{H_{ut}(H')}{H}\mathcal{B}(H_{ut}(H')) + \mu\overline{\theta}\left[H' + \sum_{t=1}^{\infty} (\mu\beta)^t H_{ut}(H')\right].$$

The residents' optimal choice of H' can be characterized by maximizing with respect to H'. The first order condition is

$$\begin{array}{c} \left[\frac{\mathcal{B}(H') + H'\mathcal{B}'(H')}{H} + \sum_{t=1}^{\infty} \beta^t \frac{\mathcal{B}(H_{ut}(H')) + H_{ut}(H')\mathcal{B}'(H_{ut}(H'))}{H} H'_{ut}(H')\right] \\ +\mu\overline{\theta}\left(1 + \sum_{t=1}^{\infty} (\mu\beta)^t H'_{ut}(H')\right) = 0, \end{array} \tag{40}$$

which is analogous to (33). The term in the square brackets is the change in the current price of housing resulting from a marginal increase in H'. It will be negative at an optimal solution. Note that it is partially determined by the response of future residents to a marginal increase in H' which is measured by $H'_{ut}(H')$. The remaining term represents the marginal benefit of an increase in H' which reflects the increase in the surplus from living in the community which is not capitalized into the price.

We show in the on-line Appendix that (40) implies that the solution housing rule $H_u(H)$ satisfies the condition

$$(1-H')\overline{\theta}+S+2sH'-C(1-\beta)-\underline{u}=\overline{\theta}(H'-\mu H)\left[1+\sum_{t=1}^{\infty}(\mu\beta)^t\,H'_{ut}(H')\right]. \tag{41}$$

This can be usefully compared with (34) which defines the commitment housing level.

We next conjecture that the solution housing rule will take a linear form so that $H'_u(H)=\gamma$ for some $\gamma\in[0,1)$. Then, condition (41) implies that

$$H_u(H)=\frac{\left(\overline{\theta}+S-C(1-\beta)-\underline{u}\right)(1-\mu\beta\gamma)}{2\left(\overline{\theta}-s\right)-\mu\beta\gamma\left(\overline{\theta}-2s\right)}+\left(\frac{\mu\overline{\theta}}{2\left(\overline{\theta}-s\right)-\mu\beta\gamma\left(\overline{\theta}-2s\right)}\right)H, \tag{42}$$

which confirms the conjecture. It must then be the case that

$$\gamma=\frac{\mu\overline{\theta}}{2\left(\overline{\theta}-s\right)-\mu\beta\gamma\left(\overline{\theta}-2s\right)}, \tag{43}$$

or, equivalently, that

$$\mu\beta\left(\overline{\theta}-2s\right)\gamma^2-2\left(\overline{\theta}-s\right)\gamma-\mu\overline{\theta}=0. \tag{44}$$

We show in the on-line Appendix that the solution to this quadratic equation which lies in the relevant range is

$$\gamma=\frac{\mu\overline{\theta}}{\overline{\theta}+\sqrt{(1-\mu^2\beta)\overline{\theta}\left(\overline{\theta}-2s\right)+s^2}-s}=\frac{\mu\overline{\theta}}{\overline{\theta}+\xi}, \tag{45}$$

where ξ is as defined in the text. Substituting (45) into (42) reveals that the solution housing rule is given by

$$H_u(H)=\frac{\left(\overline{\theta}+S-C(1-\beta)-\underline{u}\right)\xi}{\left(\overline{\theta}-2s\right)\left(\overline{\theta}+\xi\right)}+\frac{\mu\overline{\theta}}{\overline{\theta}+\xi}H. \tag{46}$$

The associated price rule is

$$P_u(H)=C+\frac{H_u(H)}{H}\mathcal{B}(H_u(H))+\sum_{t=1}^{\infty}\beta^t\frac{H_{ut}(H_u(H))}{H}\mathcal{B}(H_{ut}(H_u(H))), \tag{47}$$

where the sequence $\langle H_{ut}(H)\rangle_{t=1}^{\infty}$ is defined as above but uses the rule $H_u(H)$ in (46). The value function is given by

$$V_u(H)=P_u(H)-\mu\overline{\theta}\left[1-H'_u(H)+\sum_{t=1}^{\infty}(\mu\beta)^t\,(1-H_{ut}(H'_u(H)))\right]+\frac{\underline{u}}{1-\beta}. \tag{48}$$

9.2.3 The solution to the unconstrained problem satisfies Lemma 1

By Lemma 1, the solution to the unconstrained problem $\{H_u(H), P_u(H), V_u(H)\}$ as defined in (46), (47), and (48) can be used to create a solution to problem (18) if it satisfies certain conditions. These are that the housing rule $H_u(H)$ is increasing on the interval $[H_0, 1]$, that $H_u(H_0) > H_0$, that $H_u(1) < 1$, and that there exists a unique housing level $\widetilde{H} \geq H^*$ such that $H_u(\widetilde{H}) = \widetilde{H}$. These conditions are satisfied. The intercept of $H_u(H)$ is positive and its slope is less than one. The associated housing level $\widetilde{H}$ is given by

$$\widetilde{H} = \frac{\left(\overline{\theta} + S - C(1-\beta) - \underline{u}\right)\xi}{\left(\overline{\theta} - 2s\right)\left(\overline{\theta}(1-\mu) + \xi\right)} = H^{**}. \tag{49}$$

As noted in the text, $H^{**} \geq H^*$. It is clear that the solution to problem (18) associated with $\{H_u(H), P_u(H), V_u(H)\}$ as defined in (36), (37), and (38), is exactly $\{H'(H), P(H), V(H)\}$ as defined in (20), (21), and (22). It follows that $\{H'(H), P(H), V(H)\}$ as defined in (20), (21), and (22) is an equilibrium

9.2.4 The equilibrium has the claimed properties

As in the text, define the sequence $\langle H_t(H)\rangle_{t=1}^{\infty}$ inductively as follows: $H_1(H) = H'(H)$ and $H_t(H) = H'(H_{t-1}(H))$ for all $t \geq 2$, where the function $H'(H)$ is as defined in (20). Then $H_t(H_0)$ is the housing level that will prevail at the beginning of period t. Given the assumption that $H_0 < H^{**}$ and the properties of $H'(H)$, it is clear that $H_1(H_0) > H_0$ and that, for all $t \geq 2$, $H_t(H_0) > H_{t-1}(H_0)$. Moreover, $\lim_{t\to\infty} H_t(H_0) = H^{**}$. Thus, the housing stock converges asymptotically to H^{**}.

From (21), the price in period $t \geq 1$ is

$$P(H_t(H_0)) = C + \sum_{z=1}^{\infty} \beta^{z-1} \frac{H_{t+z}(H_0)}{H_t(H_0)} \mathcal{B}(H_{t+z}(H_0)).$$

This price converges to

$$C + \frac{\mathcal{B}(H^{**})}{1-\beta}.$$

To understand the claims about taxation and subsidization, note that $H^{**} < (>) H^e$ as $s < (>)$ $\overline{\theta}(1-\mu)/\left[\mu\sqrt{(1-\beta)} + 1 - \mu\right]$. In the former case, given that the housing stock is increasing, the net private benefit $\mathcal{B}(H)$ is positive in all periods and hence price is always above construction cost. Thus, new construction is taxed in each period. Moreover, because the product $H\mathcal{B}(H)$ is decreasing in H under Assumption 1, it follows that housing prices are decreasing. In the latter case, while the price of housing will eventually be below construction cost, it will start out above if H_0 is sufficiently small simply because $H_1(H_0)$ will be less than H^e (since $H'(0) < H^e$). Thus, new construction is eventually subsidized, but will initially be taxed if the initial housing stock is sufficiently small. At first glance, it is not obvious that the price of housing will be decreasing over time in this case because, if new construction is subsidized, a larger population may reduce the per-capita cost of such subsidies. To complete the proof, we therefore develop the expression for price in (21) by substituting in the equilibrium housing levels and show that the resulting expression is always decreasing in H. This analysis can be found in the on-line Appendix.

9.3 Proof of Proposition 7

9.3.1 The two conditions

We begin by presenting two conditions that are sufficient for there to exist a stalled equilibrium with a steady state housing level $\widetilde{H}$.

Lemma 2 *Let $\widetilde{H} \in [H_0, H^{**})$. Suppose that $\widetilde{H}$ satisfies the conditions*

$$(1-\widetilde{H})\overline{\theta} + S + 2s\widetilde{H} - C(1-\beta) - \underline{u} \geq \overline{\theta}(\widetilde{H} - \mu H_0), \tag{50}$$

and

$$C + \frac{\mathcal{B}(\widetilde{H})}{1-\beta} - P(\widetilde{H}) \geq \mu\overline{\theta}\left[\sum_{t=1}^{\infty}(\mu\beta)^{t-1} H_t(\widetilde{H}) - \frac{\widetilde{H}}{1-\mu\beta}\right], \tag{51}$$

where the sequence $\left\langle H_t(\widetilde{H})\right\rangle_{t=1}^{\infty}$ is defined inductively as: $H_1(\widetilde{H}) = H'(\widetilde{H})$ and $H_t(\widetilde{H}) = H'(H_{t-1}(\widetilde{H}))$ for all $t \geq 2$ using the housing rule $H'(H)$ in (20) and $P(H)$ is the price rule in (21). Then, there exists a solution of problem (18) of the form described in (23)-(25) with steady state housing level $\widetilde{H}$.

Proof of Lemma 2 Let $\widetilde{H} \in [H_0, H^{**})$ satisfy conditions (50) and (51). Then we need to show that $\{H'(H), P(H), V(H)\}$ as defined in (23), (24), and (25), solve problem (18). There are three steps to the proof. **Step 1** involves showing that given the price rule $P(H)$ in (24) and the value function $V(H)$ in (25) the housing rule $H'(H)$ in (23) solves problem (18). Substituting the price constraint into the objective function, this amounts to showing that $H'(H)$ solves the problem

$$\max_{H'}\left\{\begin{array}{c} C + \frac{H'}{H}\mathcal{B}(H') + \beta\frac{H'}{H}(P(H') - C) - \mu\overline{\theta}(1-H') + \mu\beta\left(V(H') - P(H')\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} \\ s.t.\ H' \geq H \end{array}\right\}.$$

We already know that $H'(H)$ is optimal on the interval $(\widetilde{H}, 1]$, so we just need to focus on the interval $[H_0, \widetilde{H}]$.

Consider then some $H \in [H_0, \widetilde{H}]$. Then, $H'(H) = \widetilde{H} \geq H$. Substituting in for $P(H'(H))$ and $V(H'(H))$ the equilibrium payoff is

$$C + \frac{\widetilde{H}}{H(1-\beta)}\mathcal{B}(\widetilde{H}) - \frac{\mu\overline{\theta}}{1-\mu\beta}(1-\widetilde{H}) + \frac{\underline{u}}{1-\beta}.$$

There are two types of deviations to consider. The first is to some smaller housing level $H' \in [H, \widetilde{H})$. Given equilibrium play, the consequences of such a choice would be to just delay the increase to $\widetilde{H}$ until the next period. The payoff from such a deviation would therefore be:

$$C + \frac{H'\mathcal{B}(H') + \frac{\beta}{1-\beta}\widetilde{H}\mathcal{B}(\widetilde{H})}{H} - \mu\overline{\theta}\left[(1-H') + \frac{\mu\beta}{1-\mu\beta}(1-\widetilde{H})\right] + \frac{\underline{u}}{1-\beta}.$$

The derivative of this payoff with respect to H' is

$$\frac{H'\mathcal{B}'(H') + \mathcal{B}(H')}{H} + \mu\overline{\theta}.$$

If this derivative is positive on $[H, \widetilde{H})$, such a deviation cannot be profitable. Recall that $H'\mathcal{B}'(H') + \mathcal{B}(H')$ is decreasing in H'. Thus, this derivative is positive on $[H, \widetilde{H})$ if

$$\frac{\widetilde{H}\mathcal{B}'(\widetilde{H}) + \mathcal{B}(\widetilde{H})}{H} + \mu\overline{\theta} \geq 0.$$

Using (17), this condition amounts to

$$(1-\widetilde{H})\overline{\theta} + S + 2s\widetilde{H} - C(1-\beta) - \underline{u} \geq \overline{\theta}(\widetilde{H} - \mu H).$$

Observe that this condition will be satisfied for any $H \in [H_0, \widetilde{H}]$ if (50) is satisfied. Accordingly, deviation to some smaller housing level $H' \in [H, \widetilde{H})$ is not profitable.

The second type of deviation is to some larger housing level $H' \in (\widetilde{H}, 1)$. First, consider deviations of this form in which $H' \leq H^{**}$. Given the equilibrium play following this deviation, the payoff from it can be written as

$$C + \frac{H'\mathcal{B}(H') + \sum_{t=1}^{\infty}\beta^t H_t(H')\mathcal{B}(H_t(H'))}{H} - \mu\overline{\theta}\left[1 - H' + \sum_{t=1}^{\infty}(\mu\beta)^t\left(1 - H_t(H')\right)\right] + \frac{\underline{u}}{1-\beta}$$

where the sequence $\langle H_t(H)\rangle_{t=1}^{\infty}$ is defined using the housing rule (23) or, since the two rules coincide on the interval $(\widetilde{H}, 1]$, the housing rule (20). Thus, to show that the deviation is not profitable, we need to show that

$$
\begin{aligned}
& C+\frac{\widetilde{H}}{H(1-\beta)}\mathcal{B}(\widetilde{H})-\frac{\mu\overline{\theta}}{1-\mu\beta}(1-\widetilde{H}) \\
\geq \quad & C+\frac{H'\mathcal{B}(H')+\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))}{H}-\mu\overline{\theta}\left[1-H'+\sum_{t=1}^{\infty}(\mu\beta)^{t}\left(1-H_t(H')\right)\right]+\frac{\underline{u}}{1-\beta},
\end{aligned}
$$

or, equivalently, that

$$
\frac{1}{H}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H})-H'\mathcal{B}(H')-\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))\right]\geq\mu\overline{\theta}\left[H'+\sum_{t=1}^{\infty}(\mu\beta)^{t}H_t(H')-\frac{\widetilde{H}}{1-\mu\beta}\right].
$$

Now, condition (51) implies that

$$
C+\frac{\widetilde{H}}{\widetilde{H}(1-\beta)}\mathcal{B}(\widetilde{H})-\frac{\mu\overline{\theta}}{1-\mu\beta}(1-\widetilde{H})+\frac{\underline{u}}{1-\beta}\geq V(\widetilde{H}),
$$

where $V(H)$ is the value function in (22). Moreover, since $V(H)$ is the value function for problem (18), we must have that

$$
V(\widetilde{H})\geq C+\frac{H'\mathcal{B}(H')+\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))}{\widetilde{H}}-\mu\overline{\theta}\left[1-H'+\sum_{t=1}^{\infty}(\mu\beta)^{t}\left(1-H_t(H')\right)\right]+\frac{\underline{u}}{1-\beta}.
$$

Thus, we have that

$$
\frac{1}{\widetilde{H}}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H})-H'\mathcal{B}(H')-\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))\right]\geq\mu\overline{\theta}\left[H'+\sum_{t=1}^{\infty}(\mu\beta)^{t}H_t(H')-\frac{\widetilde{H}}{1-\mu\beta}\right].
$$

Since H' and $H_t(H')$ exceed $\widetilde{H}$, it follows that

$$
\begin{aligned}
& \frac{1}{\widetilde{H}}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H})-H'\mathcal{B}(H')-\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))\right] \\
\geq \quad & \mu\overline{\theta}\left[H'+\sum_{t=1}^{\infty}(\mu\beta)^{t}H_t(H')-\frac{\widetilde{H}}{1-\mu\beta}\right]>0.
\end{aligned}
$$

Since $H\leq\widetilde{H}$, it therefore follows that

$$
\begin{aligned}
& \frac{1}{H}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H})-H'\mathcal{B}(H')-\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))\right] \\
\geq \quad & \frac{1}{\widetilde{H}}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H})-H'\mathcal{B}(H')-\sum_{t=1}^{\infty}\beta^{t}H_t(H')\mathcal{B}(H_t(H'))\right] \\
\geq \quad & \mu\overline{\theta}\left[H'+\sum_{t=1}^{\infty}(\mu\beta)^{t}H_t(H')-\frac{\widetilde{H}}{1-\mu\beta}\right],
\end{aligned}
$$

as required.

A similar argument can be applied to rule out deviations to a larger housing level in which $H' > H^{**}$. The details can be found in the on-line Appendix.

Step 2 involves establishing that, given the housing rule in (23), the price rule in (24) satisfies

$$P(H) = C + \frac{H'(H)}{H}\mathcal{B}(H'(H)) + \beta\frac{H'(H)}{H}(P(H'(H)) - C).$$

Given what we already know, it suffices to check this for $H \in [H_0, \widetilde{H}]$. This follows easily from (23) and (24).

Step 3 involves establishing that, given the housing rule in (23) and the price rule in (24), the value function in (25) satisfies:

$$V(H) = P(H) - \mu\overline{\theta}(1 - H'(H)) + \mu\beta\left(V(H'(H)) - P(H'(H))\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u}.$$

Again, given what we already know, it suffices to check this for $H \in [H_0, \widetilde{H}]$. Again, this follows easily from (23), (24), and (25). ■

9.3.2 Implications of the two conditions

The implications of condition (50) are straightforward. Under our Assumptions it will be satisfied if

$$\widetilde{H} \leq \frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{2(\overline{\theta} - s)} + \frac{\mu\overline{\theta}}{2(\overline{\theta} - s)}H_0. \tag{52}$$

Importantly for the Proposition, it is easy to see from (16) that condition (50) will be satisfied strictly by $\mathcal{H}(H_0)$ and hence for any smaller housing levels than $\mathcal{H}(H_0)$.

The implications of condition (51) are less obvious. It is convenient to define the function on the interval $[H_0, H^{**}]$

$$\varphi(H) \equiv C + \frac{\mathcal{B}(H)}{1-\beta} + \mu\overline{\theta}\frac{H}{1-\mu\beta} - \left[P(H) + \mu\overline{\theta}\left(\sum_{t=1}^{\infty}(\mu\beta)^{t-1}H_t(H)\right)\right], \tag{53}$$

where the sequence $\left\langle H_t(\widetilde{H})\right\rangle_{t=1}^{\infty}$ is defined using the housing rule $H'(H)$ in (20). The second condition will be satisfied if $\varphi(\widetilde{H}) \geq 0$. Intuitively, $\varphi(\widetilde{H})$ represents the difference in payoffs experienced by the residents if the current housing level is $\widetilde{H}$ and they remain with this level rather than choosing the optimal deviation $H'(\widetilde{H})$ where $H'(H)$ is the housing rule defined in (20).

It is clear that $\varphi(H^{**}) = 0$ and, assuming $\mu < 1$, that $\varphi(H^*) > 0$. The latter follows because, with current housing level H^*, remaining with H^* is the solution that the residents would choose with commitment (recall that $\mathcal{H}(H^*)$ is just H^*). However, in the equilibrium, when $\mu < 1$, the housing level increases gradually to H^{**}. We show in the on-line Appendix that $lim_{H \searrow 0}\varphi(H) = -\infty$ and that $\varphi(H)$ is concave. It follows from all this that, when $\mu < 1$, there exists a unique housing level $\underline{H} < H^*$ such that $\varphi(\underline{H}) = 0$. This housing level has the property that $\varphi(H)$ is non-negative in the range $[\underline{H}, H^{**}]$ and is negative everywhere else. Thus, condition (51) will be satisfied for all $\widetilde{H} \in [\underline{H}, H^{**}]$.

9.3.3 Completing the proof

We claim that $\underline{H}$ has the property described in the Proposition. We know that $\underline{H} < H^*$. Let $H_0 \in (\underline{H}, H^*)$ and take any $\widetilde{H} \in (H_0, \mathcal{H}(H_0))$. Note that $\widetilde{H}$ satisfies condition (50) since $\widetilde{H} < \mathcal{H}(H_0)$. It also satisfies condition (51) because $\widetilde{H}$ belongs to the interval $[\underline{H}, H^*]$. Thus, with initial housing stock H_0 there exist an equilibrium of the form described in (23)-(25) with steady state housing level $\widetilde{H}$. Moreover, by construction, this steady state housing level is less than that which would be chosen in the initial residents' optimal plan $\mathcal{H}(H_0)$.

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Appendix (For Online Publication)

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Community Development with Externalities and Corrective Taxation

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10 On-line Appendix

10.1 Micro-founding the surplus function

Let g denote the public spending of the community. Assume this is financed by a uniform tax, so that each resident pays g/H if the population is H. Let b denote the benefits of public spending for a resident. These depend on both g and the number of residents H, so that $b = b(g, H)$. The function b is increasing in g but decreasing in H because of crowding. The surplus a resident gets from living in the community is

$$U(H, b) - \frac{g}{H}$$

where the function $U(H, b)$ reflects both the utility from the benefits of public spending but also any independent cost or benefit of having a higher population. The latter could result from congestion or from the benefits of social interactions. Clearly, $U(H, b)$ is increasing in b. It could be increasing or decreasing in H depending on the setting. Let $g^*(H)$ denote the optimal level of public spending when the community has H residents; that is,

$$g^*(H) = \arg\max_g U(H, b(g, H)) - \frac{g}{H}.$$

Then, under the assumption that public spending is set optimally, the surplus obtained by each resident when there are H residents is

$$S(H) = U(H, b(g^*(H), H)) - \frac{g^*(H)}{H}.$$

We are interested in the conditions under which it can be written in the form

$$S(H) = S + sH,$$

where S is a positive constant and s is a positive or negative constant.

In general, we know that for all $H \in [H_0, 1]$ we have that

$$S(H) = S(H_0) + S'(\widetilde{H})(H - H_0)$$

for some $\widetilde{H} \in [H_0, H]$. If $S'(\widetilde{H})$ is constant over the relevant range of housing levels, we have that

$$S(H) = \left[S(H_0) - S'(\widetilde{H})H_0\right] + S'(\widetilde{H})H$$

and we can define S to equal $S(H_0) - S'(\widetilde{H})H_0$ and s to equal $S'(\widetilde{H})$.

Under what circumstances is the derivative $S'(\widetilde{H})$ going to be constant over the relevant range? We know that

$$S'(H) = \frac{\partial U(H, b(g^*(H), H))}{\partial H} + \frac{\partial U(H, b(g^*(H), H))}{\partial b}\frac{\partial b(g^*(H), H)}{\partial H} + \frac{g^*(H)}{H^2}. \tag{54}$$

Assume that

$$U(H, b) = \kappa H + b,$$

where ξ can be positive or negative. This implies that the non-public spending related costs or benefits of having a higher population are linear over the relevant range and independent of the level of public benefits. It ensures that the first term in (54) is constant. This just leaves us with the second and third terms.

We can identify three sets of circumstances under which sum of the second and third terms in (54) will be constant. The first is when public spending goes to finance a public service or a publicly-provided private good the costs of which are just proportional to the number of residents. Specifically, assume that

$$b(g, H) = \widetilde{b}(\frac{g}{H})$$

where $\widetilde{b}$ is increasing and strictly concave. This implies that the optimal level of spending is proportional to the population

$$g^*(H) = H\widetilde{b}'^{-1}(1).$$

Then,

$$\frac{\partial U(H, b(g^*(H), H))}{\partial b}\frac{\partial b(g^*(H), H)}{\partial H} + \frac{g^*(H)}{H^2} = -\frac{\widetilde{b}'(\frac{g^*(H)}{H})g^*(H)}{H^2} + \frac{g^*(H)}{H^2} = 0,$$

where the second equality follows after using the first order condition for optimal provision. This implies that

$$S'(H) = \kappa.$$

Note that the assumption that local governments provide a public service is common in models in the Tiebout tradition.

The second circumstance is when public spending goes to finance a public good and the preferences have a particular form. Specifically,

$$b(g, H) = b_0\sqrt{g}.$$

The fact that there is no crowding reflects the fact that we are dealing with a public good. In this case,

$$g^*(H) = H^2 \left(\frac{b_0}{2}\right)^2$$

Thus, it is the case that

$$S'(H) = \kappa + \frac{b_0^2}{4}.$$

The third circumstance is when public spending goes to finance an impure public good so that there is some crowding. Specifically, assume that

$$b(g, H) = b_0 \frac{(\frac{g}{H^\alpha})^\sigma}{\sigma},$$

where σ is between 0 and 1 and α is between 0 and 1. In this formulation, α measures the congestibility of the public good and σ represents the concavity of the public good benefit function. With this specification,

$$g^*(H) = H^{\frac{1-\alpha\sigma}{1-\sigma}} b_0^{\frac{1}{1-\sigma}}$$

Thus,

$$\frac{g^*(H)}{H^\alpha} = \frac{H^{\frac{1-\alpha\sigma}{1-\sigma}} b_0^{\frac{1}{1-\sigma}}}{H^\alpha} = b_0^{\frac{1}{1-\sigma}} H^{\frac{1-\alpha}{1-\sigma}}$$

and

$$\frac{g^*(H)}{H} = \frac{H^{\frac{1-\alpha\sigma}{1-\sigma}} b_0^{\frac{1}{1-\sigma}}}{H} = b_0^{\frac{1}{1-\sigma}} H^{\sigma\left(\frac{1-\alpha}{1-\sigma}\right)}$$

This implies that

$$\begin{aligned} b(g^*(H), H) - \frac{g^*(H)}{H} &= b_0 \frac{\left[b_0^{\frac{1}{1-\sigma}} H^{\frac{1-\alpha}{1-\sigma}}\right]^\sigma}{\sigma} - b_0^{\frac{1}{1-\sigma}} H^{\sigma\left(\frac{1-\alpha}{1-\sigma}\right)} \\ &= \frac{1-\sigma}{\sigma} [b_0]^{\frac{1}{1-\sigma}} \cdot H^{\frac{\sigma(1-\alpha)}{1-\sigma}}. \end{aligned}$$

This in turn implies that

$$S(H) = \kappa H + \frac{1-\sigma}{\sigma} [b_0]^{\frac{1}{1-\sigma}} \cdot H^{\frac{\sigma(1-\alpha)}{1-\sigma}}.$$

In the case in which

$$\sigma(1-\alpha) = 1 - \sigma \Rightarrow \sigma = \frac{1}{2-\alpha}$$

we have that $S'(H)$ is constant.

10.2 Omitted details of the Proof of Proposition 6

10.2.1 Proof of Lemma 1

There are three steps to the proof. **Step 1** involves showing that given the price rule $P(H)$ in (37) and the value function $V(H)$ in (38), the housing rule $H'(H)$ in (36) solves problem (18). Substituting the price constraint into the objective function of problem (18), this amounts to showing that $H'(H)$ solves

$$\max_{H'} \left\{ \begin{array}{c} C + \frac{H'}{H}\mathcal{B}(H') + \beta\frac{H'}{H}(P(H') - C) - \mu\overline{\theta}(1 - H') + \mu\beta\left(V(H') - P(H')\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} \\ s.t.\ H' \geq H \end{array} \right\}.$$

This requires showing that for any $H \in [H_0, 1]$, it is the case that $H'(H) \geq H$ and that there does not exist an alternative housing level $\hat{H}$ satisfying the constraint that $\hat{H} \geq H$ which generates a higher value of the objective function.

First, let $H \in [\widetilde{H}, 1]$. Then we have that $H'(H) = H$. Moreover, for any alternative housing level $\hat{H} > H$, the equilibrium play of future residents would be to simply keep housing at $\hat{H}$. Thus, the problem faced by the residents is identical to that in the commitment case and, since $\widetilde{H} \geq H^*$, we know that the optimal strategy is just to maintain the current housing stock.

Second, let $H \in [H_0, \widetilde{H})$. Then we have that $H'(H) = H_u(H)$. Note that the assumptions on $H_u(H)$ together with the fact that $H \in [H_0, \widetilde{H})$ imply that $H_u(H) > H$. Deviation to some housing level $\hat{H} \in [H, \widetilde{H}]$ cannot increase the value of the objective function because in this region $V(H)$ and $V_u(H)$ coincide. Deviation to some $\hat{H} \in [\widetilde{H}, 1]$ cannot be profitable either. To understand why, note that once such a deviation occurs, the equilibrium play of future residents would be to simply keep housing at $\hat{H}$. Since

$$P(\hat{H}) - C = \frac{\mathcal{B}(\hat{H})}{1-\beta}$$

and

$$V(\hat{H}) - P(\hat{H}) = -\frac{\mu\overline{\theta}}{1-\mu\beta}(1-\hat{H}) + \frac{\underline{u}}{1-\beta},$$

the problem of optimally choosing such a deviation amounts to

$$\max_{\hat{H}} \left\{ \begin{array}{c} C + \frac{\hat{H}}{H}\mathcal{B}(\hat{H}) + \beta\frac{\hat{H}}{H}\frac{\mathcal{B}(\hat{H})}{1-\beta} - \mu\overline{\theta}(1-\hat{H}) + \mu\beta\left(-\frac{\mu\overline{\theta}}{1-\mu\beta}(1-\hat{H}) + \frac{\underline{u}}{1-\beta}\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} \\ s.t.\ \hat{H} \geq \widetilde{H} \end{array} \right\}$$

or, equivalently,

$$\max_{\hat{H}} \left\{ \begin{array}{c} \frac{\hat{H}}{H}\frac{\mathcal{B}(\hat{H})}{1-\beta} + \frac{\mu\overline{\theta}\hat{H}}{1-\mu\beta} \\ \\ s.t.\ \hat{H} \geq \widetilde{H} \end{array} \right\}.$$

This problem has the same objective function as in problem (32). In the proof of Proposition 3, we established this objective function is concave. Moreover, for $H < H^*$ it has a maximum at $\mathcal{H}(H) < \widetilde{H}$ and for $H \geq H^*$ it has a maximum at $H < \widetilde{H}$. It follows that the objective function will have the highest value at the corner: $\widetilde{H}$. Hence such a deviation cannot be profitable because, as we have just shown, $H_u(H)$ provides a higher payoff than $\widetilde{H}$. This concludes **Step 1**.

Step 2 involves establishing that, given the housing rule in (36), the price rule in (37) satisfies

$$P(H) = C + \frac{H'(H)}{H}\mathcal{B}(H'(H)) + \beta\frac{H'(H)}{H}(P(H'(H)) - C).$$

For $H \geq \widetilde{H}$ this is trivially true. For $H < \widetilde{H}$ this is also true because $H'(H) < \widetilde{H}$ and hence $P(H)$, $H'(H)$ and $P(H'(H))$ coincide with their respective counterparts $P_u(H)$, $H_u(H)$ and $P_u(H_u(H))$, for which the equation above holds as part of the solution to the unconstrained problem.

Step 3 involves establishing that, given the housing rule in (36) and the price rule in (37), the value function in (38) satisfies:

$$V(H) = P(H) - \mu\overline{\theta}(1 - H'(H)) + \mu\beta\left(V(H'(H)) - P(H'(H))\right) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u}.$$

For $H \geq \widetilde{H}$ this is trivially true. For $H < \widetilde{H}$ this is also true because $H'(H) < \widetilde{H}$ and hence $V(H)$, $P(H)$, $H'(H)$, $V'(H'(H))$ and $P(H'(H))$ coincide with their respective counterparts $V_u(H), P_u(H)$, $H_u(H), V_u(H_u(H))$ and $P_u(H_u(H))$.

10.2.2 Proof of (41)

To establish (41), we first show that

$$\sum_{t=1}^{\infty}\beta^t\frac{\mathcal{B}(H_{ut}(H')) + H_{ut}(H')\mathcal{B}'(H_{ut}(H'))}{H}H'_{ut}(H') = -\frac{\mu\overline{\theta}\beta H'\sum_{t=1}^{\infty}(\mu\beta)^{t-1}H'_{ut}(H')}{H}. \tag{55}$$

To see this, observe that the first order condition for $H_{u1}(H')$ implies that

$$\begin{aligned}
&\frac{\mathcal{B}(H_{u1}(H'))+H_{u1}(H')\mathcal{B}'(H_{u1}(H'))}{H'} \\
&+\sum_{t=1}^{\infty}\beta^{t}\frac{\mathcal{B}(H_{ut+1}(H_{u1}(H')))+H_{ut+1}(H_{u1}(H'))\mathcal{B}'(H_{ut+1}(H_{u1}(H')))}{H'}H'_{ut+1}(H_{u1}(H')) \\
= \quad &-\mu\overline{\theta}\left[1+\sum_{t=1}^{\infty}(\mu\beta)^{t}H'_{ut+1}(H_{u1}(H'))\right].
\end{aligned}$$

Multiplying this through by $\beta H'_{u1}(H')$ we obtain

$$\begin{aligned}
&\beta\frac{\mathcal{B}(H_{u1}(H'))+H_{u1}(H')\mathcal{B}'(H_{u1}(H'))}{H'}H'_{u1}(H') \\
&+\sum_{t=1}^{\infty}\beta^{t+1}\frac{\mathcal{B}(H_{ut+1}(H_{u1}(H')))+H_{ut+1}(H_{u1}(H'))\mathcal{B}'(H_{ut+1}(H_{u1}(H')))}{H'}H'_{ut+1}(H_{u1}(H'))H'_{u1}(H') \\
= \quad &-\mu\overline{\theta}\beta\left[H'_{u1}(H')+\sum_{t=1}^{\infty}(\mu\beta)^{t}H'_{ut+1}(H_{u1}(H'))H'_{u1}(H')\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{t=1}^{\infty}\beta^{t}\frac{\mathcal{B}(H_{ut}(H'))+H_{ut}(H')\mathcal{B}'(H_{ut}(H'))}{H'}H'_{ut}(H')=-\mu\overline{\theta}\beta\left[H'_{u1}(H')+\sum_{t=1}^{\infty}(\mu\beta)^{t}H'_{ut+1}(H_{u1}(H'))H'_{u1}(H')\right] \\
= \quad &-\mu\overline{\theta}\beta\left[\sum_{t=0}^{\infty}(\mu\beta)^{t}H'_{ut+1}(H')\right] \\
= \quad &-\mu\overline{\theta}\beta\left[\sum_{t=1}^{\infty}(\mu\beta)^{t-1}H'_{ut}(H')\right].
\end{aligned}$$

This follows from the fact that for all $t\geq 2$

$$H'_{ut}(H_{u1}(H'))H'_{u1}(H')=H'_{ut}(H').$$

Thus, (55) holds.

Using (55), the impact on the housing price of marginally raising H' can be written as

$$-\frac{\mathcal{B}(H')+H'\mathcal{B}'(H')}{H}+\frac{\mu\overline{\theta}\beta H'\sum_{t=1}^{\infty}(\mu\beta)^{t-1}H'_{ut}(H')}{H}.$$

The impact on the discounted sum of future housing stocks of marginally raising H' is

$$\mu\overline{\theta}\left[1+\sum_{t=1}^{\infty}(\mu\beta)^{t}H'_{ut}(H')\right].$$

The first order condition is therefore that

$$\begin{aligned}
-\mathcal{B}(H') - H'\mathcal{B}'(H') &= \mu\overline{\theta}\left[H + H\sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H') - \beta H'\sum_{t=1}^{\infty}(\mu\beta)^{t-1} H'_{ut}(H')\right] \\
&= \overline{\theta}\left[\mu H + \mu H\sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H') - H'\sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H')\right].
\end{aligned}$$

This implies that

$$-\mathcal{B}(H') - H'\mathcal{B}'(H') - \overline{\theta}H' = \overline{\theta}\left[\mu H + \mu H\sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H') - H' - H'\sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H')\right],$$

or

$$\mathcal{B}(H') + H'\mathcal{B}'(H') + \overline{\theta}H' = \overline{\theta}(H' - \mu H)\left[1 + \sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H')\right].$$

Since

$$\mathcal{B}(H') + H'\mathcal{B}'(H') + \overline{\theta}H' = (1 - H')\overline{\theta} + S + 2sH' - C(1-\beta) - \underline{u},$$

this implies that

$$(1 - H')\overline{\theta} + S + 2sH' - C(1-\beta) - \underline{u} = \overline{\theta}(H' - \mu H)\left[1 + \sum_{t=1}^{\infty}(\mu\beta)^t H'_{ut}(H')\right],$$

which is (41).

10.2.3 Solving (43)

Equation (43) is a quadratic with solutions

$$\frac{\left(\overline{\theta} - s\right) \pm \sqrt{s^2 + (1-\mu^2\beta)\left(\overline{\theta}^2 - 2s\overline{\theta}\right)}}{\mu\beta\left(\overline{\theta} - 2s\right)}.$$

We claim that the positive root exceeds 1. To see this, note that

$$\begin{aligned}
&\overline{\theta} - s + \sqrt{s^2 + (1-\mu^2\beta)\left(\overline{\theta}^2 - 2s\overline{\theta}\right)} - \mu\beta\left(\overline{\theta} - 2s\right) \\
=\ &\overline{\theta}(1-\mu\beta) - s(1 - 2\mu\beta) + \sqrt{s^2 + (1-\mu^2\beta)\left(\overline{\theta}^2 - 2s\overline{\theta}\right)}.
\end{aligned}$$

Divide through by $\overline{\theta}$ and denote $\hat{s} = s/\overline{\theta}$ to get

$$(1-\mu\beta) - \hat{s}(1 - 2\mu\beta) + \sqrt{\hat{s}^2 + (1-\mu^2\beta)(1 - 2\hat{s})} =$$

$$(1-\mu\beta)(1-\hat{s}) + \sqrt{\hat{s}^2 + (1-\mu^2\beta)(1 - 2\hat{s})} - \hat{s}.$$

Since $1-\hat{s}>0$ and $1-2\hat{s}>0$, the expression above is positive. Hence, the positive root exceeds 1.

The negative root is

$$\gamma=\frac{\overline{\theta}-s-\sqrt{s^2+(1-\mu^2\beta)\left(\overline{\theta}^2-2s\overline{\theta}\right)}}{\mu\beta\left(\overline{\theta}-2s\right)}.$$

This implies that

$$\begin{aligned}2\left(\overline{\theta}-s\right)-\mu\beta\gamma\left(\overline{\theta}-2s\right) &= 2\left(\overline{\theta}-s\right)-\left(\overline{\theta}-s-\sqrt{s^2+(1-\mu^2\beta)\left(\overline{\theta}^2-2s\overline{\theta}\right)}\right)\\ &= \overline{\theta}-s+\sqrt{s^2+(1-\mu^2\beta)\left(\overline{\theta}^2-2s\overline{\theta}\right)}.\end{aligned}$$

Thus, given (43), it is also the case that

$$\gamma=\frac{\mu\overline{\theta}}{\overline{\theta}+\sqrt{(1-\mu^2\beta)\overline{\theta}\left(\overline{\theta}-2s\right)+s^2}-s}. \tag{56}$$

Using the definition of ξ in the text, we can write this more compactly as

$$\gamma=\frac{\mu\overline{\theta}}{\overline{\theta}+\xi}. \tag{57}$$

10.2.4 The price of housing is decreasing

To prove that the price of housing is decreasing we show that when $H\leq H^{**}$ $P(H)$ can be written as follows:

$$P(H)=C+\bar{\theta}\left(\frac{\pi_{-1}}{H}+\pi_0+\pi_1 H\right), \tag{58}$$

where $\{\pi_{-1},\pi_0,\pi_1\}$ are coefficients which depend on underlying parameters of the model such that $\pi_{-1}>0$ and $\pi_1<0$. Specifically, it is the case that

$$\begin{aligned}\pi_{-1} &= \frac{\left[(1-\hat{u})\hat{a}-(1-\hat{s})\hat{a}^2\right]}{1-\beta}-\frac{\left[(1-\hat{u})-2(1-\hat{s})\hat{a}\right]}{1-\beta b}b\hat{a}-\frac{(1-\hat{s})}{1-\beta b^2}b^2\hat{a}^2\\ \pi_0 &= \frac{\left[(1-\hat{u})-2(1-\hat{s})\hat{a}\right]}{1-\beta b}b+\frac{2(1-\hat{s})}{1-\beta b^2}b^2\hat{a}\\ \pi_1 &= -\frac{(1-\hat{s})}{1-\beta b^2}b^2\end{aligned}$$

where

$$\hat{s} \equiv s/\bar{\theta}$$

$$\hat{u} \equiv -\frac{S-C(1-\beta)-\underline{u}}{\bar{\theta}}$$

$$a \equiv \frac{\left(\bar{\theta}+S-C(1-\beta)-\underline{u}\right)\xi}{\left(\bar{\theta}-2s\right)\left(\bar{\theta}+\boldsymbol{\xi}\right)} = \frac{(1-\hat{u})\frac{\xi}{\bar{\theta}}}{(1-2s)\left(1+\frac{\xi}{\bar{\theta}}\right)}$$

$$b \equiv \frac{\mu\bar{\theta}}{\bar{\theta}+\xi} = \frac{\mu}{1+\frac{\xi}{\bar{\theta}}}$$

$$\hat{a} \equiv \frac{a}{1-b}.$$

We derive (58) in two steps. First, note that for all $H \leq H^{**}$

$$H'(H) = a + bH.$$

Second, from Step 2 of the proof of Lemma 1 we have that

$$H\left(P(H) - C\right) = H'(H)\mathcal{B}(H'(H)) + \beta H'(H)(P(H'(H)) - C).$$

Denote $Q(H) \equiv H\left(P(H) - C\right)/\bar{\theta}$ and substitute for $H'(H)\mathcal{B}(H'(H))$ to write the expression above as

$$Q(H) = (1-\hat{u})(H'(H)) + (\hat{s}-1)\left(H'(H)\right)^2 + \beta Q(H'(H)).$$

The method of undetermined coefficients yields

$$Q(H) = \pi_{-1} + \pi_0 H + \pi_1 H^2,$$

where $\{\pi_{-1}, \pi_0, \pi_1\}$ can be solved for directly. This implies (58).

Returning to $P(H)$, since $\hat{s} < 1$, we have that $\pi_1 < 0$. It remains to show that $\pi_{-1} > 0$. After some algebraic manipulations we have that

$$\pi_{-1} = (1-\hat{s})a\frac{1}{1-\beta}\frac{1}{1-\beta b}\frac{1-\underline{u}}{1-2\hat{s}}\left(\frac{1-2\hat{s}}{1-\hat{s}} - \frac{1+\beta b}{1-\beta b^2}\frac{\frac{\xi}{\bar{\theta}}}{1+\frac{\xi}{\bar{\theta}}}\right).$$

Hence to determine the sign π_{-1} we need to determine the sign of the expression in brackets on the right hand side. After more algebra, we have that

$$\frac{1+\beta b}{1-\beta b^2}\frac{\frac{\xi}{\bar{\theta}}}{1+\frac{\xi}{\bar{\theta}}} = 1 + \frac{(1-\beta\mu)\left(\hat{s} - \sqrt{(1-\mu^2\beta)(1-2\hat{s}) + \hat{s}^2}\right) - (1-\mu^2\beta)}{-2\hat{s} + 2(1-\hat{s})\sqrt{(1-\mu^2\beta)(1-2\hat{s}) + \hat{s}^2} + 2(1-\mu^2\beta)(1-\hat{s}) + 2\hat{s}^2}$$

To establish that $\pi_{-1} > 0$ it remains to show that

$$\frac{1-2\hat{s}}{1-\hat{s}} = 1 - \frac{\hat{s}}{1-\hat{s}} > 1 - \frac{(1-\beta\mu)\left(\hat{s} - \sqrt{(1-\mu^2\beta)(1-2\hat{s}) + \hat{s}^2}\right) - (1-\mu^2\beta)}{-2\hat{s} + 2(1-\hat{s})\sqrt{(1-\mu^2\beta)(1-2\hat{s}) + \hat{s}^2} + 2(1-\mu^2\beta)(1-\hat{s}) + 2\hat{s}^2},$$

or that

$$-\frac{\hat{s}}{1-\hat{s}} > -\frac{(1-\beta\mu)\left(\hat{s}-\sqrt{(1-\mu^2\beta)(1-2\hat{s})+\hat{s}^2}\right)-(1-\mu^2\beta)}{-2\hat{s}+2(1-\hat{s})\sqrt{(1-\mu^2\beta)(1-2\hat{s})+\hat{s}^2}+2(1-\mu^2\beta)(1-\hat{s})+2\hat{s}^2}.$$

With some algebra, it can be shown that the inequality above is equivalent to

$$\hat{s} < 1/2\left(1-\mu\beta+\mu\beta\frac{\Theta}{\sqrt{\Theta(1-2\hat{s})+\hat{s}^2}+\Theta-\hat{s}}\right),$$

where $\Theta = 1-\mu^2\beta$.

For negative values of s the inequality holds trivially. Hence, since Assumption 1 implies that $2\hat{s} < 1$, it remains to show that for $\hat{s} \in [.5(1-\mu\beta), .5)$ the inequality above holds. After some algebra, it can be written as

$$1 < \Theta\left(\frac{\mu\beta}{2\hat{s}-(1-\mu\beta)}-1\right)^2 + 2\hat{s}\frac{\mu\beta}{2\hat{s}-(1-\mu\beta)}.$$

Note that for $s \in [.5(1-\mu\beta), .5)$

$$2\hat{s}\frac{\mu\beta}{2\hat{s}-(1-\mu\beta)} > 1 \Leftrightarrow 2\hat{s}(\mu\beta-1) > \mu\beta-1 \Leftrightarrow 2\hat{s}(1-\mu\beta) < (1-\mu\beta),$$

which concludes the proof that π_{-1} is positive.

10.3 Omitted details of the Proof of Proposition 7

10.3.1 Deviations to a housing level $H' > H^{**}$

Given the equilibrium play following this deviation, the payoff from it can be written as

$$C + \frac{H'\mathcal{B}(H')}{H(1-\beta)} - \frac{\mu\overline{\theta}}{1-\mu\beta}(1-H') + \frac{\underline{u}}{1-\beta}.$$

Thus, to show that the deviation is not profitable, we need to show that

$$C + \frac{\widetilde{H}}{H(1-\beta)}\mathcal{B}(\widetilde{H}) - \frac{\mu\overline{\theta}}{1-\mu\beta}(1-\widetilde{H}) \geq C + \frac{H'\mathcal{B}(H')}{H(1-\beta)} - \frac{\mu\overline{\theta}}{1-\mu\beta}(1-H') + \frac{\underline{u}}{1-\beta},$$

or, equivalently, that

$$\frac{1}{H}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H}) - \frac{H'}{1-\beta}\mathcal{B}(H')\right] \geq \mu\overline{\theta}\left[\frac{H'}{1-\mu\beta} - \frac{\widetilde{H}}{1-\mu\beta}\right].$$

Now, (51) implies that

$$C + \frac{\widetilde{H}}{\widetilde{H}(1-\beta)}\mathcal{B}(\widetilde{H}) - \frac{\mu\overline{\theta}}{1-\mu\beta}(1-\widetilde{H}) + \frac{\underline{u}}{1-\beta} \geq V(\widetilde{H}).$$

Moreover, by Lemma 1, $V(\widetilde{H})$ solves problem (18), implying that

$$V(\widetilde{H}) \geq C + \frac{H'\mathcal{B}(H')}{\widetilde{H}(1-\beta)} - \frac{\mu\overline{\theta}}{1-\mu\beta}(1-H') + \frac{\underline{u}}{1-\beta}.$$

Thus, we have that

$$\frac{1}{\widetilde{H}}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H}) - \frac{H'}{1-\beta}\mathcal{B}(H')\right] \geq \mu\overline{\theta}\left[\frac{H'}{1-\mu\beta} - \frac{\widetilde{H}}{1-\mu\beta}\right].$$

Since H' exceeds $\widetilde{H}$, it follows that

$$\frac{1}{\widetilde{H}}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H}) - \frac{H'}{1-\beta}\mathcal{B}(H')\right] \geq \mu\overline{\theta}\left[\frac{H'}{1-\mu\beta} - \frac{\widetilde{H}}{1-\mu\beta}\right] > 0.$$

Since $H \leq \widetilde{H}$, it therefore follows that

$$\begin{aligned} & \frac{1}{H}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H}) - \frac{H'}{1-\beta}\mathcal{B}(H')\right] \geq \frac{1}{\widetilde{H}}\left[\frac{\widetilde{H}}{1-\beta}\mathcal{B}(\widetilde{H}) - \frac{H'}{1-\beta}\mathcal{B}(H')\right] \\ \geq \;& \mu\overline{\theta}\left[\frac{H'}{1-\mu\beta} - \frac{\widetilde{H}}{1-\mu\beta}\right], \end{aligned}$$

as required.

10.3.2 Properties of $\varphi(H)$

It only remains to show that $\lim_{H\searrow 0}\varphi(H) = -\infty$ and that $\varphi(H)$ is concave. Note that $\sum_{t=1}^{\infty}(\mu\beta)^{t-1}H_t(H)$ is linear in H because $H'(H)$ and, and hence, all $H_t(H)$ are linear in H. Furthermore, $C + \frac{\mathcal{B}(H)}{1-\beta} + \mu\overline{\theta}\frac{H}{1-\mu\beta}$ is also linear in H. From (58), we know that for $H \leq H^{**}$

$$P(H) = C + \bar{\theta}\left[\frac{\pi_{-1}}{H} + \pi_0 + \pi_1 H\right].$$

This implies that

$$\varphi(H) = -\bar{\theta}\frac{\pi_{-1}}{H} + \phi_0 + \phi_1 H,$$

where ϕ_0 and ϕ_1 are constants that depend on underlying parameters of the model. Since π_{-1} is positive, we have that $\varphi''(H) = -\frac{2\pi_{-1}}{H^3} < 0$ and hence $\varphi(H)$ is concave. Clearly, $\lim_{H\searrow 0}\varphi(H) = -\infty$.

10.4 Proof of Proposition 8

Part (i) of the Proposition follows from the discussion in the text. For Part (ii) consider first the case in which $H_0 < H^*$. Again, denote $\hat{s} = s/\overline{\theta}$ and $\hat{u} = \frac{\underline{u}-S-C(1-\beta)}{\overline{\theta}}$. After some straightforward manipulations, one can show that

$$
\begin{aligned}
W^e(s) &= \frac{\int_{(1-H^e)\overline{\theta}}^{\overline{\theta}} \theta \frac{d\theta}{\overline{\theta}} + H^e\left(S + sH^e\right) + \underline{u}\left(1 - H^e\right)}{1-\beta} - C(H^e - H_0) \\
&= \overline{\theta}\frac{\frac{1}{2}\left(\frac{1-\hat{\underline{u}}}{1-\hat{s}}\right)^2 + \frac{\underline{u}}{\overline{\theta}}}{1-\beta} + CH_0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
W^c(s) &= \frac{\int_{(1-\mathcal{H}(H_0))\overline{\theta}}^{\overline{\theta}} \theta \frac{d\theta}{\overline{\theta}} + \mathcal{H}(H_0)(S + s\mathcal{H}(H_0)) + \underline{u}(1-\mathcal{H}(H_0))}{1-\beta} - C(\mathcal{H}(H_0) - H_0) \\
&= \overline{\theta}\frac{\frac{2\hat{s}-1}{8(1-\hat{s})^2}\left(1-\hat{u}+\frac{\mu(1-\beta)}{(1-\mu\beta)}H_0\right)^2 + \frac{1}{2(1-\hat{s})}\left(1-\hat{u}+\frac{\mu(1-\beta)}{(1-\mu\beta)}H_0\right)(1-\hat{u}) + \frac{\underline{u}}{\overline{\theta}}}{1-\beta} + CH_0.
\end{aligned}
$$

Hence,

$$
W^c(s) - W^e(s) = \overline{\theta}\left[\frac{3-2\hat{s}}{8}\left(\frac{1-\hat{u}}{1-\hat{s}}\right)^2 + \frac{(1-\hat{u})}{4\left(1-\hat{s}\right)^2}\frac{\mu(1-\beta)}{(1-\mu\beta)}H_0 + \frac{2\hat{s}-1}{8\left(1-\hat{s}\right)^2}\left(\frac{\mu(1-\beta)}{(1-\mu\beta)}H_0\right)^2 - \frac{1}{2}\left(\frac{1-\hat{u}}{1-\hat{s}}\right)^2\right].
$$

This is a quadratic expression in H_0. A convenient way to analyze this is to denote $x = \frac{\mu(1-\beta)}{(1-\mu\beta)}H_0\frac{1}{1-\hat{\underline{u}}}$. Then we have that

$$
W^c(s) - W^e(s) = \overline{\theta}\frac{(1-\hat{u})^2}{(1-\hat{s})^2}\left[-\frac{1+2\hat{s}}{8} + \frac{x}{4} + \frac{2\hat{s}-1}{8}x^2\right].
$$

This implies that the sign of $W^c(s) - W^e(s)$ is determined by the sign of the following expression:

$$
-\left(1+2\hat{s}\right) + 2x + \left(2\hat{s}-1\right)x^2.
$$

This is a parabola with a negative lead coefficient and the following roots:

$$
x_1 = \frac{1+2\hat{s}}{1-2\hat{s}} \text{ and } x_2 = 1.
$$

It follows that for $x \in (x_1, x_2)$ $W^c(s) - W^e(s)$ will be positive. It will be negative for x outside this range. Substituting back for x, we have that $W^c(s) - W^e(s)$ will be positive if and only if

$$
\frac{\mu(1-\beta)}{(1-\mu\beta)}H_0\frac{1}{1-\hat{\underline{u}}} \in \left(\frac{\overline{\theta}+2s}{\overline{\theta}-2s}, 1\right).
$$

Substituting for $\hat{\underline{u}}$ and re-arranging the expression above, we have that $W^c(s) - W^e(s)$ will be positive if and only if

$$
H_0 \in \left(\left(\frac{1-\mu\beta}{\mu(1-\beta)}\right)\left(\frac{\overline{\theta}+S+C(1-\beta)-\underline{u}}{\overline{\theta}}\right)\frac{\overline{\theta}+2s}{\overline{\theta}-2s}, \left(\frac{1-\mu\beta}{\mu(1-\beta)}\right)\left(\frac{\overline{\theta}+S+C(1-\beta)-\underline{u}}{\overline{\theta}}\right)\right).
$$

Note that since $s < 0$

$$\begin{aligned}\left(\frac{1-\mu\beta}{\mu(1-\beta)}\right)\left(\frac{\overline{\theta}+S+C(1-\beta)-\underline{u}}{\overline{\theta}}\right) &> \frac{\overline{\theta}+S+C(1-\beta)-\underline{u}}{\overline{\theta}-s} = H^e \\ &> H^o \\ &\geq H^*.\end{aligned}$$

Since $H_0 < H^*$, a necessary condition for it to be possible for $W^c(s) - W^e(s)$ to be positive, is that

$$\left(\frac{1-\mu\beta}{\mu(1-\beta)}\right)\left(\frac{\overline{\theta}+S+C(1-\beta)-\underline{u}}{\overline{\theta}}\right)\frac{\overline{\theta}+2s}{\overline{\theta}-2s} < H^* = \frac{\overline{\theta}+S-C(1-\beta)-\underline{u}}{\overline{\theta}(1+\frac{1-\mu}{1-\mu\beta})-2s},$$

which, after some manipulations, is equivalent to.

$$\overline{\theta}^2(1-\mu) + s\overline{\theta}\left(1-\mu\beta\right) - 2s^2\left(1-\mu\beta\right) < 0.$$

The left hand side of this expression is increasing when s is negative. Consider the quadratic equation

$$s^2 2\left(1-\mu\beta\right) - s\overline{\theta}\left(1-\mu\beta\right) - \overline{\theta}^2(1-\mu) = 0.$$

It has roots

$$\frac{\overline{\theta}\left[(1-\mu\beta) \pm \sqrt{(1-\mu\beta)^2 + 8(1-\mu)\left(1-\mu\beta\right)}\right]}{4\left(1-\mu\beta\right)}.$$

Hence, if

$$s > \overline{\theta}\left(\frac{(1-\mu\beta) - \sqrt{(1-\mu\beta)^2 + 8(1-\mu)\left(1-\mu\beta\right)}}{4\left(1-\mu\beta\right)}\right),$$

welfare is higher with free entry. Rearranging this yields (27). If this inequality is not satisfied, welfare is higher under commitment if and only if

$$H_0 \in \left(\frac{1-\mu\beta}{\mu(1-\beta)}\left(\frac{\overline{\theta}-s}{\overline{\theta}}\right)\left(\frac{\overline{\theta}+2s}{\overline{\theta}-2s}\right)H^e, H^*\right).$$

This is the condition reported in the proposition, once it is recognized that $H^e = \frac{\overline{\theta}-2s}{\overline{\theta}-s}H^o$.

Now consider the case in which $H_0 \geq H^*$. In this case, we have that:

$$W^e(s) = \bar{\theta}\frac{\left(\hat{s}-\frac{1}{2}\right)H^{e2} + H^e(1-\hat{u}) + \frac{\underline{u}}{\overline{\theta}}}{1-\beta} + CH_0,$$

and

$$W^c(s) = \bar{\theta}\frac{\left(\hat{s}-\frac{1}{2}\right)H_0^2 + H_0(1-\hat{u}) + \frac{\underline{u}}{\overline{\theta}}}{1-\beta} + CH_0.$$

Hence,

$$W^c(s) - W^e(s) = \frac{\bar{\theta}}{(1-\beta)}\left(1-2\hat{s}\right)\left(H_0 - \frac{1-\hat{u}}{1-\hat{s}}\right)\left[\frac{(1-\hat{u})}{1-2\hat{s}} - \frac{1-\hat{u}}{2(1-\hat{s})} - \frac{H_0}{2}\right]$$

Since $H_0 - \frac{1-\hat{u}}{1-\hat{s}} < 0$, this implies that $W^c(s) - W^e(s) < 0$ when

$$\frac{2(1-\hat{u})}{1-2\hat{s}} - \frac{1-\hat{u}}{(1-\hat{s})} > H_0,$$

or

$$\frac{1-\hat{u}}{(1-2\hat{s})\left(1-\hat{s}\right)} = \frac{1}{1-2\hat{s}}H^e > H_0.$$

When $\hat{s} < 0$ the condition above is satisfied for all H_0 less than $\frac{1}{1-2\hat{s}}H^e$. Equivalently, when $s < 0$ commitment is better than free entry if

$$H_0 > \frac{\overline{\theta}}{\overline{\theta} - 2s}H^e.$$

Since $H_0 \geq H^*$, commitment will be better than free entry for all H_0, if $H^* \geq \frac{\overline{\theta}}{\overline{\theta}-2s}H^e$. Substituting for H^* and H^e this condition can be written as

$$\frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{\overline{\theta}(1 + \frac{1-\mu}{1-\mu\beta}) - 2s} \geq \frac{\overline{\theta}}{\overline{\theta} - 2s}\left(\frac{\overline{\theta} + S - C(1-\beta) - \underline{u}}{\overline{\theta} - s}\right).$$

Re-arranging we can write this as

$$2s^2 - s\overline{\theta} - \overline{\theta}^2\frac{1-\mu}{1-\mu\beta} \geq 0.$$

The equation

$$2s^2 - s\overline{\theta} - \overline{\theta}^2\frac{1-\mu}{1-\mu\beta} = 0,$$

has roots

$$s = \frac{\overline{\theta}\left(1 \pm \sqrt{1 + 8\frac{1-\mu}{1-\mu\beta}}\right)}{4}.$$

Thus, commitment will be better than free entry for all H_0 if

$$s \leq \frac{\overline{\theta}\left(1 - \sqrt{1 + 8(\frac{1-\mu}{1-\mu\beta})}\right)}{4},$$

which occurs when condition (27) is not satisfied. If condition (27) is satisfied, then welfare is higher with commitment if and only if $H_0 > \frac{\overline{\theta}}{\overline{\theta}-2s}H^e$. This is the condition reported in the Proposition once it is recognized that $H^e = \frac{\overline{\theta}-2s}{\overline{\theta}-s}H^o$.

10.5 Commitment versus free entry for higher μ

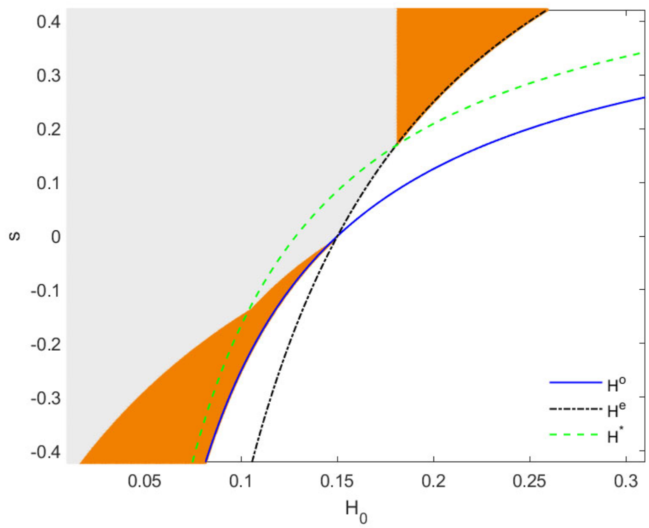

Figure 6: Commitment versus Free Entry, $\mu = .99$

10.6 Gradual equilibrium versus best stalled equilibrium

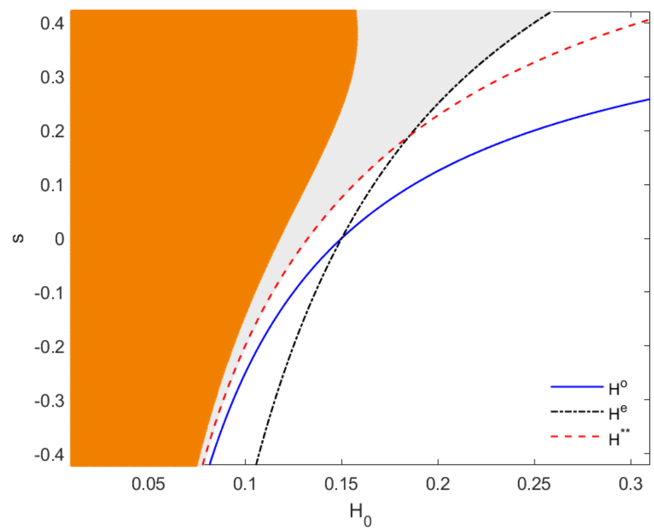

Figure 7: Gradual Equilibrium versus Best Stalled Equilibrium

The gray (light) and orange (dark) shaded areas in Figure 7 represent (H_0, s) pairs which satisfy Assumptions 1 and 2 and for which gradual and stalled equilibria exist (which requires $H_0 < H^{**}$). The orange shaded area represents combinations for which the gradual equilibrium dominates the best stalled equilibrium (i.e., the one with the highest steady state housing level $\widetilde{H}$).